Integer Programming Formulation of the Problem of Generating Milton Babbitt's All-partition Arrays

Tsubasa Tanaka[1], Brian Bemman[2], and David Meredith[2]

[1] STMS Lab : IRCAM, CNRS, UPMC, Paris, France

tsubasa.tanaka@ircam.fr

[2] The Music Informatics and Cognition Group, AD:MT, Aalborg University, Denmark

{bb,dave}@create.aau.dk

1. Objective

- Our aim is to generate the 12-tone compositional structure developed by Milton Babbitt known as an ***all-partition array***.
- An all-partition array is a covering of an $I \times J$ pitch-class matrix by a collection of subsets, each containing an ***aggregate*** and each represented by a distinct ***integer partition*** of 12.

2. Our method

- We formulate the problem of generating an all-partition array using ***integer programming (IP)***, a powerful programming paradigm in which problems are described with discrete linear equations and/or inequalities.
- We introduce the use of *overlaps* between subsets, instead of *insertions*, which allow us to define the generation of an all-partition array as a ***set-covering problem (SCP)***.

3. Introducing Overlaps

- An all-partition array's matrix contains fewer elements than does its required collection of subsets. Therefore, additional pitch classes must be found.
- Original construction:

11	4	3	**3**	5	9	10	1	8	2	0	7	6
6	7	**7**	0	2	8	1	10	9	5	3	4	11
5	6	11	1	7	0	9	8	4	2	3	10	
2	9	10	**10**	8	4	3	0	5	11	1	6	7
0	5	4	6	10	11	2	9	3	1	8	7	
1	8	9	7	3	2	11	4	10	0	5	6	

Figure 1: Subsets represented by the integer partitions $[3,3,2,2,1,1]$ and $[3,3,3,3]$. Insertions of additional pitch classes make the matrix irregular.

V.S.

- Our formulation:

11	4	3	5	9	10	1	8	2	0	7	6
6	7	0	2	8	1	10	9	5	3	4	11
5	6	11	1	7	0	9	8	4	2	3	10
2	9	10	8	4	3	0	5	11	1	6	7
0	5	4	6	10	11	2	9	3	1	8	7
1	8	9	7	3	2	11	4	10	0	5	6

Figure 2: These same subsets with overlaps allowing for the matrix to remain regular.

4. Subsets C_k & Constraints

- A binary variable $x_{i,j,k}$ indicates whether or not (i,j) in the matrix belongs to a kth subset, C_k.
- C_k is then the set of all (i,j) where $x_{i,j,k} = 1$.

11	4	3	5
6	7	0	2
5	6	11	1
2	9	10	8
0	5	4	6
1	8	9	7

(a) kth subset.

1	1	1	0
1	1	0	0
1	0	0	0
1	1	1	0
1	0	0	0
1	1	0	0

(b) All $x_{i,j,k} = 1$ for kth subset.

(1,1)	(2,2)	(3,3)
(2,1)	(2,2)	
(3,1)		
(4,1)	(4,2)	(4,3)
(5,1)		
(6,1)	(6,2)	

(c) C_k.

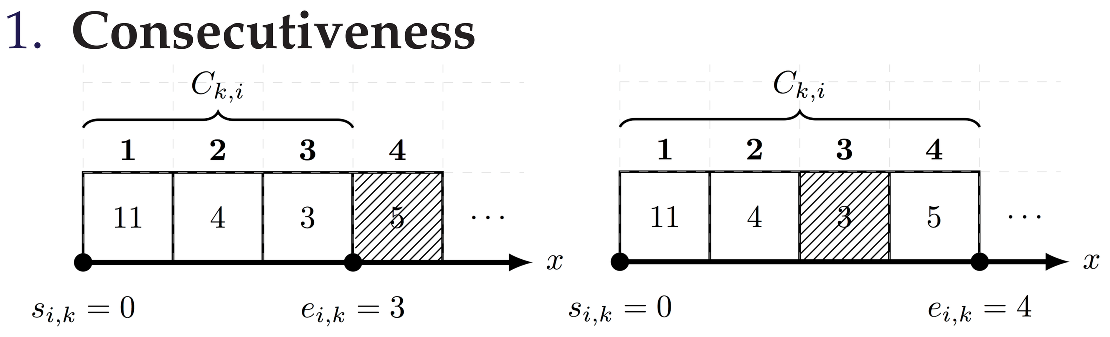
Figure 3

Solution Constraints

1. Consecutiveness of row-elements in C_k.
2. Containment of 12 pitch classes in C_k.
3. Covering of all matrix elements by 58 C_k.
4. Restrictions on overlaps in contiguous C_k.
5. Distinctness of C_k integer partitions.

5. IP Formulation

1. **Consecutiveness**



The elements of $C_{k,i}$ are found in a range from a starting point s_k to ending point e_k:

$$\forall i \in [1,I], \forall k \in [1,K],\ 0 \le s_{i,k} \le e_{i,k} \le J.$$

Any element of $C_{k,i}$ is located to the lefthand side of $e_{i,k}$:

$$\forall i \in [1,I], \forall j \in [1,J], \forall k \in [1,K],\ \ j \cdot x_{i,j,k} \le e_{i,k},$$

Any element of $C_{k,i}$ is located to the righthand side of $s_{i,k}$:

$$\forall i \in [1,I], \forall j \in [1,J], \forall k \in [1,K],\ \ J - s_{i,k} \ge (J+1-j) \cdot x_{i,j,k},$$

The length of $C_{k,i}$ is exactly $e_{i,k} - s_{i,k}$:

$$\forall i \in [1,I], \forall k \in [1,K],\ \sum_{j=1}^{J} x_{i,j,k} = e_{i,k} - s_{i,k}.$$

5. IP Formulation (cont.)

2. **Containment**

Exactly 12 distinct pitch classes are contained in C_k:

$$\forall p \in [0,11], \forall k \in [1,K],\ \sum_{i=1}^{I}\sum_{j=1}^{J} B^p_{i,j} \cdot x_{i,j,k} = 1,$$

where B^P contains all (i,j) in the matrix corresponding to pitch class, p.

3. **Covering**

Each element in the matrix is covered at least once:

$$\forall i \in [1,I], \forall j \in [1,J],\ \sum_{k=1}^{K} x_{i,j,k} \ge 1.$$

4. **Restrictions on overlaps**

Contiguous C_k subsets can not overlap by more than one location in $C_{k,i}$:

$$\forall i \in [1,I], \forall k \in [2,K],\ e_{i,k-1} \le e_{i,k},$$

$$\forall i \in [1,I], \forall k \in [2,K],\ e_{i,k-1} - 1 \le s_{i,k} \le e_{i,k-1},$$

5. **Distinctness**

Variable $y_{i,k,l}$ allows us to convert the horizontal lengths of $C_{k,i}$ into column sums, where:

$$\forall i \in [1,I], \forall k \in [1,K],\ e_{i,k} - s_{i,k} = \sum_{l=1}^{L} y_{i,k,l},$$

$$\forall i \in [1,I], \forall k \in [1,K], \forall l \in [2,L],\ \ y_{i,k,l-1} \ge y_{i,k,l},$$

such that $\forall i \in [1,I]$, $y_{i,k,l}$ becomes Figure 3(b) and $\sum_{i=1}^{I} y_{i,k,l} = [6,4,2,0,0,0,0,0,0,0,0,0]$. $P_{n,l}$ $(1 \le l \le L)$ specifies the shape of C_k as these column sums. Binary variable $z_{k,n}$ indicates whether C_k takes the shape $P_{n,l}$ or not:

$$\forall k \in [1,K], \forall n \in [1,N], \forall l \in [1,L],\ \ P_{n,l} \cdot z_{k,n} \le \sum_{i=1}^{I} y_{i,k,l}.$$

All C_k correspond to a distinct integer partition:

$$\forall n \in [1,N],\ \sum_{k=1}^{K} z_{k,n} = 1.$$

8. Solving smaller problems

Solving for the entire problem (i.e., 6×96 matrix and 58 subsets), proved too difficult. Therefore, we solved for smaller problems consisting of the first J columns of the original matrix, a number of subsets equal to $(J+2)/2$, and 12 overlaps.

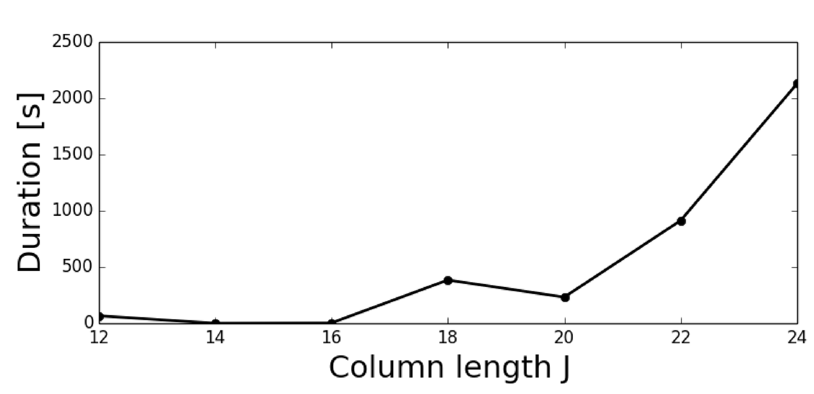
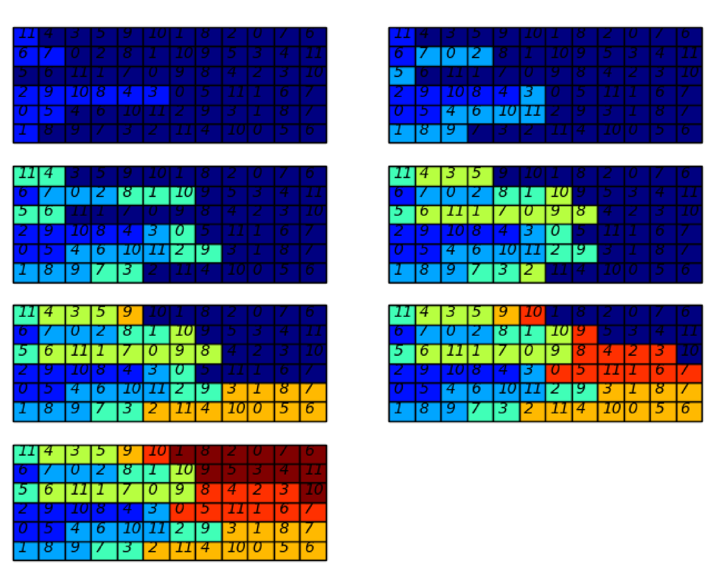


Figure 4: Solving time for smaller problems.

The figure below shows one solution for a smaller matrix ($J = 12$, 7 subsets). In future work, we hope to combine such smaller solutions to form a complete solution to the entire problem.



9. Acknowledgements/Refs

The work of Tsubasa Tanaka reported in this paper was supported by JSPS Postdoctoral Fellowships for Research Abroad. The work of Brian Bemman and David Meredith was carried out as part of the project Lrn2Cre8, which acknowledges the financial support of the Future and Emerging Technologies (FET) programme within the Seventh Framework Programme for Research of the European Commission, under FET grant number 610859.

[1] Babbitt, M. Since Schoenberg. Perspectives of New Music, 12(1/2):3–28 (1973).

[2] Bazelow, A.R., Brickle, F.: A Partition Problem Posed by Milton Babbitt. Perspectives of New Music 14, 2, 280–293 (1976)

[3] Mead, A.: An Introduction to the Music of Milton Babbitt. Princeton University Press, Princeton, NJ (1994)

[4] Bemman, B. and Meredith, D.: Generating Milton Babbitt's all-partition arrays, Journal of New Music Research, 45, 2 (2016)

[5] Tsubasa Tanaka T. and Fujii K.: Melodic Pattern Segmentation of Polyphonic Music as a Set Partitioning Problem, Proceedings of ICMM (2014)