MIPS
A Formal Language for the Mathematical Investigation of Pitch Systems

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Chapter 1

Introduction to *MIPS* and the genus representation of octave equivalence

1.1 Introduction

MIPS is a mathematical formal language devised by the author for investigating the structural properties of scales, pitch systems and their associated notational systems.[1] The complete current specification of *MIPS* is given in Chapter 4. *MIPS* has been implemented as a computer program written in Common Lisp.

MIPS models the way that pitch information is represented within Western staff notation. In fact, it models a whole class of pitch notation systems that contains the Western staff notation system as one of its members. In this sense, *MIPS* mathematically models and generalises the pitch representation system used in Western staff notation.

MIPS is based on four representations of octave equivalence: *chroma equivalence*, *morph equivalence*, *chromamorph equivalence* and *genus equivalence*. Chroma equivalence is essentially identical to the concept of pitch-class equivalence used by Babbitt ([Bab65]), Forte ([For73]), Rahn ([Rah80]), Morris ([Mor87]) and many others. The *MIPS* concept of a *morph* is basically the same as Brinkman's concept of *name class* ([Bri90, 124–126]). The *MIPS* concept of a *chromamorph* is closely related to both Brinkman's *binomial representation* ([Bri90, 128]) and the representation of octave equivalence used by Agmon ([Agm89, 11], [Agm96, 44]). *Genus equivalence* is a new representation of octave equivalence invented by the author which provides a correct model of the traditional tonal concept of octave equivalence. That is, two pitches are genus equivalent if and only if they are an integer number of perfect octaves apart. Genus equivalence can also be generalised to any other pitch system without first having to specify which sets in that pitch system correspond to the diatonic sets in the Western tonal system.

Chroma equivalence is not a particularly good model of the traditional tonal concept of octave equivalence. The three pitches in Figure 1.1 are octave equivalent in the traditional tonal sense and, of course, they have the same chroma—they are therefore chroma equivalent.

The two pitches in Figure 1.2 are also chroma equivalent, but they are not octave equivalent in the traditional tonal sense because the interval between them is an augmented seventh and not an integer number of perfect octaves. So although the sounds produced when the two notes are performed in an equal-tempered system might be psycho-acoustically an octave apart, they are not 'octave equivalent' in terms of the logic of the Western tonal pitch notation system.

[1] *MIPS* stands for Mathematical Investigation of Pitch Systems.

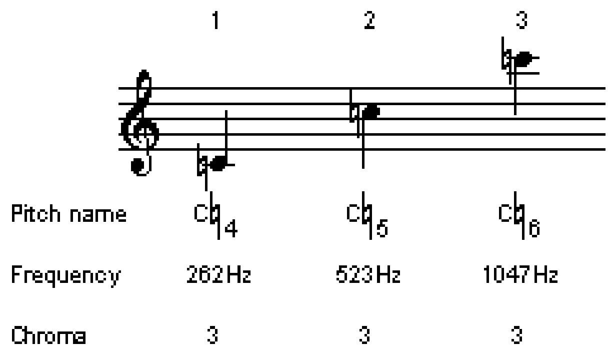


Figure 1.1: Three pitches that are chroma equivalent and ‘octave equivalent’ in the traditional tonal sense.

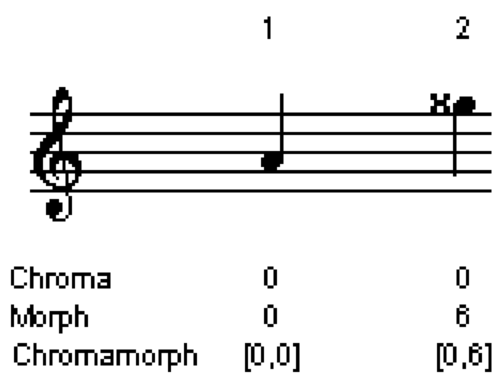
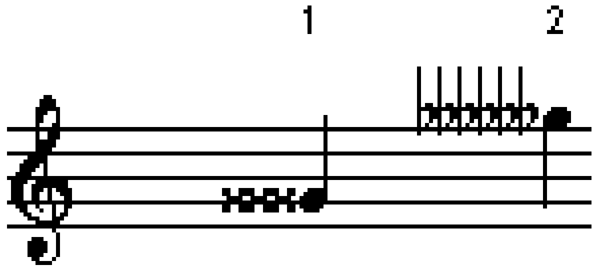


Figure 1.2: Two pitches that are chroma equivalent but not ‘octave equivalent’ in the traditional tonal sense and not chromamorph equivalent.



	1	2
Chroma	4	4
Morph	6	6
Pitch	[52,27]	[52,34]
Chromamorph	[4,6]	[4,6]
Old genus	[6,6]	[6,-6]
Genus	[16,6]	[4,6]

Figure 1.3: Two pitches that are chromamorph equivalent but not octave equivalent in the traditional tonal sense.

This demonstrates that the concept of pitch class as used by Forte ([For73]), Rahn ([Rah80]) and others, does not provide a correct model of octave equivalence within the Western tonal pitch system.

There have been a number of attempts to produce better models of the traditional tonal concept of octave equivalence. For example, Brinkman ([Bri90, 128]) and Agmon ([Agm89, 11], [Agm96, 44]) use a representation of octave equivalence that Brinkman calls a *binomial representation* which is essentially identical to the *MIPS* concept of a *chromamorph*. A chromamorph is an ordered pair of integers in which the first number represents the chroma and the second number (which in *MIPS* is called *morph* and which Brinkman calls *name class* ([Bri90, 124–126])) represents the letter-name of the note. So, in the Western tonal system, the second element in a chromamorph (that is, the morph) will have an integer value between 0 and 6, with 0 corresponding to the letter-name *A* and 6 corresponding to *G*. Similarly, in a system that uses five-note scales, the value of a morph would lie between 0 and 4.

If two notes that have the same chromamorph are defined to be *chromamorph equivalent* then it can be seen from Figure 1.2 that chromamorph equivalence is a better model of the Western tonal concept of octave equivalence than chroma equivalence—at least chromamorph equivalence correctly captures the fact that two notes an augmented seventh apart are not octave equivalent in the traditional tonal sense.

However, the two notes in Figure 1.3 *are* chromamorph equivalent but they are certainly *not* octave equivalent in the traditional Western tonal sense—the interval between them is a '12×diminished octave.'

This demonstrates that chromamorph equivalence is not a correct model of the traditional Western tonal concept of octave equivalence.

Some may dispute the claim that the two notes in Figure 1.3 are logically possible and meaningful within the Western tonal pitch notation system, but, in principle, there is no limit to the number of sharps and flats that could be placed before a note in the Western tonal staff notation system. On the upper staff in Figure 1.4 is a sequence of notes in which the interval from each note to the next note is a rising major third. Each note on the lower staff is enharmonically equivalent to the note immediately above it on the upper staff.

Figure 1.4: Demonstration of the logical possibility of multiple sharps and flats in the Western tonal pitch notation system.

The sequence of notes on the upper staff begins with an F double-sharp—a note that is encountered in tonal music as the leading note in the key of G sharp minor, the relative minor of the commonly used key of B major. As can be seen in Figure 1.4, after two consecutive leaps of a rising major third from F double sharp, we have already arrived at a note that must have three sharp symbols placed before it if it is to be notated correctly. After eleven consecutive leaps of a rising major third we are compelled to use *eight* sharps! This example illustrates the fact that a formal language that correctly represents the logic of the Western tonal system of pitch and pitch intervals must allow for pitches to have any number of sharps or flats.

In the Western pitch-naming system, a note has a *letter-name* (A to G), an *inflection* $(\ldots, \flat\flat, \flat, \natural, \sharp, \sharp\sharp, \ldots)$ and an *octave number* (for example, middle C—$C\natural_4$—has an octave number of 4 and the C above middle C ($C\natural_5$) has an octave number of 5). This naming system derives from the staff notation system which has evolved over the past four hundred years or so to be a highly effective means of notating Western tonal music. To this extent, the pitch-naming system correctly models the Western tonal pitch system. And if the octave number of a pitch-name is omitted (for example, $C\natural_4$ becomes $C\natural$), the result is a correct representation of octave equivalence within the Western tonal system.

So, if one wishes to find a correct mathematical representation of the traditional Western tonal concept of octave equivalence, one strategy might be to base a numerical representation on the traditional pitch-naming system. Such a strategy has been adopted by Cambouropoulos ([Cam96, 233], [Cam98, 49]) in his *General Pitch Interval Representation* (*GPIR*). In this system, the letter-name (A to G) is represented by an integer between 0 and 6 and the inflection (or *modifier-accidental* as Cambouropoulos calls it) is represented by an integer (0 corresponds to $\natural$, 1 corresponds to $\sharp$, -1 corresponds to $\flat$ and so on).

The row labelled 'Old genus' in Figure 1.3 shows that this representation correctly captures the fact that the two notes are not octave equivalent in the traditional sense. So this simple numeric representation of the Western tonal pitch-naming system provides a correct model of the traditional concept of octave equivalence within that system.

However, one of the motivations behind the development of *MIPS* was to produce a system that would allow one to examine the special mathematical properties of the Western tonal scales and then go on to determine if scales with similar properties exist in other systems where the octave is divided into more or less than 12 intervals. In other words, it should be possible to use *MIPS* to discover those sets within any pitch system that correspond in some significant sense to the sets associated with scales in the Western tonal system. But unfortunately, it is not possible to generalise a representation such as Cambouropoulos' to other pitch systems without first knowing which sets within that system should be considered to correspond to the diatonic sets in the Western tonal pitch system. This is because one first has to know which pitch classes correspond to the natural notes (that is, the notes that are not inflected with one or more sharp or flat symbols).

It turns out, however, that it *is* possible to devise a representation of octave equivalence that is both a correct model of the traditional tonal concept of octave equivalence *and* generalisable to any other pitch system without first specifying the sets in that system that correspond to the diatonic sets in the Western tonal system.

In *MIPS*, this model of octave equivalence is called *genus equivalence*: two pitches are genus equivalent if and only if they have the same *genus*. A genus is an ordered pair rather like a chromamorph. As in a chromamorph, the second element in the ordered pair is a morph and represents the letter-name (see the row marked 'Genus' in Figure 1.3). However, the first member of a genus is not a chroma but a *chromatic genus* which is not quite the same as chroma (see section 1.3.1 below for formal definitions of chromamorph, chromatic genus and genus). Unfortunately the fact that chromatic genus is 'not quite' chroma means that the whole theory surrounding the genus representation—the theory that defines, for example, how to transpose and invert genus sets, find powers and sums of genus intervals and so on—is rather more involved than the pitch-class set theory of Babbitt, Forte and Rahn.

In summary, *MIPS* is a formal language for investigating the mathematical properties of pitch systems and scales within those systems. It is based on four distinct mathematical representations of octave equivalence: chroma equivalence, morph equivalence, chromamorph equivalence and genus equivalence. Genus equivalence correctly models the traditional Western tonal concept of octave equivalence wherein two pitches are considered octave equivalent if and only if they are an integer number of perfect octaves apart. Furthermore, the concept of genus equivalence can be generalised to any pitch system without first having to specify which sets within that system correspond to the diatonic sets of the Western tonal system.

The rest of this section will be devoted to introducing certain basic concepts that will be used throughout this document. In section 1.2 the *MIPS* representations for the intuitive concepts of pitch system and pitch are introduced and discussed in detail. In section 1.3 the genus representation of octave equivalence is defined and the mathematical theory surrounding this representation is introduced. In section 1.4 four useful algorithms are described for

1. generating the *MIPS* pitch representation that corresponds to any given A.S.A. pitch name;
2. generating the A.S.A. pitch name that corresponds to a given *MIPS* pitch representation;
3. generating the *MIPS* pitch interval representation that corresponds to a traditional Western tonal pitch interval name (e.g. "Rising Major Third"); and
4. generating the traditional Western tonal pitch interval name that corresponds to a given *MIPS* pitch interval representation.

These algorithms employ the concepts presented in sections 1.2 and 1.3 and therefore constitute concrete examples of the kind of application that can be developed using *MIPS* concepts. Finally, in section 1.5 the main points of this chapter are summarised.

1.1.1 The relationship between pitch and frequency

In the text that follows, reference will be made on a number of occasions to 'the frequency of a pitch.' It is therefore important to understand the relationship between frequency and pitch.

The American Standards Association define the term 'pitch' to be "that attribute of auditory sensation in terms of which sounds may be ordered on a musical scale" ([Ass60]). However this definition is not satisfactory because of the ambiguity of the term "musical scale." It is proposed here that the term 'pitch' as this term is used in psycho-acoustics should be defined to mean that perceptual attribute of a simple tone (a tone with a sinusoidal waveform) that varies when the frequency of the tone is changed and the loudness is kept constant. The frequency of a simple tone can be adjusted until it is perceived to have the same pitch as some given complex tone. The pitch of the complex tone can then be *represented by* the frequency of the simple tone that has the same perceived pitch as it.

Usually, the perceived pitch of a complex harmonic tone is the same as that of a simple tone whose frequency is equal to the periodicity of the complex tone. For example, a complex tone with components at 400, 800 and 1200Hz will have a perceived pitch approximately equal to that of a simple tone with frequency 400Hz. Similarly, a complex tone with components at 1800, 2000 and 2200Hz has a pitch which is similar to that of a 200Hz simple tone.[2]

There are, however, exceptions to this simple rule. For example, Moore ([Moo89, 169]) points out that a complex tone with sine wave components at 1840, 2040 and 2240Hz has a periodicity of 40Hz. However its perceived pitch is approximately the same as that of a 204Hz sinusoid (although its pitch can also be matched to that of a sinusoid of frequency 185Hz and to that of a sinusoid of frequency 227Hz).[3]

It has also been shown that the pitch of a simple tone varies very slightly with amplitude (see [Moo89, 165]). In general, the pitch of tones below about 2000Hz decreases with increasing amplitude, while the pitch of tones above about 4000Hz increases with increasing amplitude. However, this effect is extremely small for most listeners and can be safely ignored for the purposes of this document.

Therefore, if at any point in this document it is suggested that a pitch p has a frequency f, this should be understood to mean that p is the perceived pitch of a simple tone S with frequency f. This implies that p is also the pitch of any complex tone whose pitch is perceived to be the same as that of S.

1.1.2 Some basic set-theoretical concepts

In this section and the next a number of basic set-theoretical concepts and arithmetical operations will be defined that will be used often throughout this document. An understanding of the definitions and theorems given here will make the remainder of the document much easier to follow.[4] The definitions of mathematical concepts given in this document are for the most part consistent with common mathematical usage. However there may be slight differences between the definitions given here and those that one might find in a standard mathematical dictionary such as [BB89]. These differences arise from the fact that the concepts presented here are defined for use specifically in a formal language for investigating musical pitch systems.

Definition 1 (Universal set) *An object is a well-formed* universal set *if and only if it is a well-defined collection of objects that are all distinct in some specified way.*

For example, $\{1, 2, 3, 4\}$ is a well-formed universal set but $\{1, 1, 2, 3\}$ is not because two of the objects in the collection are equal.

Definition 2 (Universal set membership) *If S is a universal set then a is an* element *or* member *of S, denoted $a \in S$, if and only if a is equal to one of the objects in S. If a is not equal to any of the objects in S then one can say that a is not an element of S and denote this fact as follows: $a \notin S$.*

[2]This is called the 'phenomenon of the missing fundamental'. For more details about this effect, see [Moo89, 167–175].

[3]For more details on the relationship between pitch and frequency, see [Moo89, 158–193].

[4]All definitions and theorems presented in the main body of this document are stated again in Chapter 4. The reference number of a definition or theorem given in the main body of the document is the same as the number of that definition or theorem in Chapter 4. In other words, the number of a definition or theorem in the main text indicates the order of appearance of the item in Chapter 4 and *not* its order of appearance in the main text.

For example, if $S = \{1, 2, 3, 4\}$ then $1 \in S$ but $5 \notin S$.

Definition 3 (Set) *An object is a well-formed* set *if and only if it is a collection of objects that are all distinct members of a single specified universal set. When written out in full, a set is enclosed within braces and the objects in the set are separated from each other by commas:*

$$S = \{s_1, s_2, \ldots\}$$

It is important to note that a set is, by definition, a collection of *distinct* objects. For example, if one defines A to be a universal set that contains all and only those integers greater than or equal to 0 and less than or equal to 10:

$$A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$

then the collection

$$C = \{1, 1, 2, 3\}$$

is *not* a well-formed set of objects in A because two of the objects in C are equal to the same object in A. However, the collection

$$B = \{1, 2, 3\}$$

is an example of a well-formed set of objects in A. Note that in this document, all the objects in a set must be members of some *single* specified universal set whereas a universal set can be any collection of distinct objects whatsoever.

Definition 4 (Ordered set) *An object is a well-formed* ordered set *if and only if it is a collection of objects (not necessarily distinct and not necessarily all from the same universal set). When written out in full, an ordered set is enclosed in square brackets and the objects in the ordered set are separated from each other by commas:*

$$S = [s_1, s_2, \ldots]$$

For example, the following are all well-formed ordered sets:

$$[4, 3, 2, 1] \qquad [4, 4, 4, 4] \qquad [3, c, \pi, G, 3]$$

If an ordered set contains exactly two objects then it can be called an *ordered pair*, if it contains three objects it can be called an *ordered triple*, if it contains four objects it can be called an *ordered quadruple* and so on.

Definition 5 (Set membership) *If S is a set or ordered set then a is an* element *or* member *of S, denoted $a \in S$, if and only if a is equal to one of the objects in S. If a is not equal to any member of S then one can say that a is not an element of S and denote this fact as follows: $a \notin S$.*

For example, if $S = \{1, 2, 3, 4\}$ then $1 \in S$ but $5 \notin S$.

Definition 6 (Set order) *If S is a set or ordered set then the* order *or* cardinality *of S, denoted $|S|$, is equal to the number of elements in S.*

For example, if $S = \{1, 2, 3, 4\}$ then $|S| = 4$ and if $S = [1, 2, 3, 4, 4, 4]$ then $|S| = 6$.

Definition 7 (Empty set) *The* empty set *is that unique set that contains no members. It is denoted $\emptyset$ or $\{\}$.*

Definition 8 (Empty ordered set) *The* empty ordered set *is that unique ordered set that contains no members. It is denoted $[\,]$.*

Definition 9 (Element of an ordered set) *If S is an ordered set,*

$$S = [s_1, s_2, \ldots s_k, \ldots]$$

then, by definition,

$$\mathrm{e}\,(S, k) = s_k$$

for all integer k such that $1 \le k \le |S|$. That is, the function $\mathrm{e}\,(S, k)$ returns the kth element of S.

For example, if $S = [1, 2, 3, 4, 3, 2, 1]$ then $\mathrm{e}\,(S, 2) = 2$, $\mathrm{e}\,(S, 4) = 4$ and $\mathrm{e}\,(S, 6) = 2$.

Definition 14 (Ordered set equality) *If S and T are two ordered sets,*

$$S = \left[s_1, s_2, \ldots s_{|S|}\right] \qquad T = \left[t_1, t_2, \ldots t_{|T|}\right]$$

then $S = T$ if and only if $|S| = |T|$ and $\mathrm{e}\,(S, k) = \mathrm{e}\,(T, k)$ for all integer values of k such that $1 \le k \le |S|$.

It is this concept of ordered set equality that distinguishes an ordered set from an arbitrary collection of objects. For two ordered sets to be equal, they must not only contain exactly the same objects, it must also be true that each object in one set is equal to the object that occupies the same position in the other set. For example,

$$[3, 2, 1] \neq [1, 2, 3]$$

Definition 15 (Set equality) *If S and T are two sets then S is equal to T, denoted $S = T$, if and only if one of the following two conditions is satisfied:*

1. *Both S and T are equal to the empty set.*

2. *Every element in S is an element in T and every element in T is an element in S.*

If S is not equal to T then this is denoted $S \neq T$.

Note that for two sets to be equal, the order in which the elements occur does not have to be the same. For example,

$$\{1, 2, 3\} = \{3, 2, 1\}$$

Definition 16 (Subset) *If S and T are two sets then S is a* subset *of T, denoted $S \subseteq T$, if and only if one of the following two conditions is satisfied:*

1. *S is the empty set.*

2. *Every element of S is also an element of T.*

If S is not a subset of T then this is denoted $S \nsubseteq T$.

For example, $\{1, 2\} \subseteq \{1, 2, 3\}$, $\emptyset \subseteq \{1, 2, 3\}$ and $\{1, 2, 3\} \subseteq \{1, 2, 3\}$.

Definition 20 (Set union) *If S and T are two sets then the* union *of S and T, denoted $S \cup T$, is the set that only contains every object that is an element of S or an element of T or an element of both S and T. That is*

$$(s \in (S \cup T)) \iff ((s \in S) \vee (s \in T))$$

For example, $\{1, 2\} \cup \{2, 3\} = \{1, 2, 3\}$.

The operation of set union is associative, as stated by the following theorem:

Theorem 21 (Associativity of set union) *The union operation on sets is* associative*. That is, if R, S and T are sets then*

$$R \cup (S \cup T) = (R \cup S) \cup T$$

The expressions $R \cup (S \cup T)$ and $(R \cup S) \cup T$ can therefore both be written

$$R \cup S \cup T$$

All the theorems given in the main body of this document are stated without proof. However, every one of these theorems is re-stated with proof in Chapter 4.

The fact that set union is associative allows for the following operation to be defined:

Definition 22 (Union of sequence of sets) *If $S_1, S_2, \ldots S_k, \ldots S_n$ is a collection of sets then, by definition,*

$$S_1 \cup S_2 \cup \ldots \cup S_k \cup \ldots \cup S_n = \bigcup_{k=1}^{n} S_k$$

Also, if S is a set, then

$$\bigcup_{s \in S} \mathrm{F}(s)$$

returns the set that only contains every object that is a member of one or more of the sets $\mathrm{F}(s)$ where s only takes any value such that $s \in S$ and where $\mathrm{F}(s)$ is some function of s that returns a set.

For example, if k only takes integer values then

$$\bigcup_{k=1}^{n} \{k\} = \{1, 2, 3, \ldots n\}$$

and if $S = \{1, 2, 3\}$ then

$$\bigcup_{k \in S} \{2k\} = \{2, 4, 6\}$$

Definition 23 (Set intersection) *If S and T are two sets then the* intersection *of S and T, denoted $S \cap T$, is the set that only contains every object s that is a member of S and a member of T:*

$$(s \in (S \cap T)) \iff ((s \in S) \wedge (s \in T))$$

For example, if $S = \{1, 2, 3, 4\}$ and $T = \{3, 4, 5, 6\}$ then $S \cap T = \{3, 4\}$.

Definition 26 (Set partition) *If S is a set then $\mathrm{P}(S)$ is a* partition *on S if and only if the following conditions are satisfied:*

1. $\mathrm{P}(S)$ *is a set.*
2. $\bigcup_{s \in \mathrm{P}(S)} s = S$.
3. $(s_1, s_2 \in \mathrm{P}(S)) \wedge (s_1 \neq s_2) \Rightarrow (s_1 \cap s_2 = \emptyset)$.

For example, if $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$ then all of the following sets are partitions on S:

$$\{\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8\}\} \qquad \{\{2, 4, 6, 8\}, \{1, 3, 5, 7\}\} \qquad \{\{1, 8, 2, 7\}, \{3, 6, 4, 5\}\}$$

1.1.3 Some arithmetical operations

In *MIPS*, much use is made of the three arithmetical operations, int, mod and div. These will now be defined.

Definition 27 (int) *The function* $\text{int}(x)$ *takes any real number* x *as its argument and returns the largest integer less than or equal to* x*. In other words,* $\text{int}(x)$ *is defined as follows:*

$$\text{int}(x) = y : (x - 1 < y \leq x) \wedge (y \in \mathbb{Z})$$

where $\mathbb{Z}$ *is the universal set of integers.*

For example, $\text{int}(3.4) = 3$ and $\text{int}(-3.4) = -4$.

Definition 33 (mod) *Given that* x *is a real number and* y *is a non-zero real number, then the binary operation* mod *is defined as follows:*

$$x \bmod y = x - y \times \text{int}\left(\frac{x}{y}\right)$$

The following table gives some examples of this operation:

$$\begin{array}{lcl}
4.3 \bmod 3 & = & 1.3 \\
4.3 \bmod -3 & = & -1.7 \\
-4.3 \bmod 3 & = & 1.7 \\
-4.3 \bmod -3 & = & -1.3 \\
4 \bmod 3 & = & 1 \\
4 \bmod -3 & = & -2 \\
-4 \bmod 3 & = & 2 \\
-4 \bmod -3 & = & -1
\end{array}$$

Definition 48 (div) *If* x *is a real number and* y *is a non-zero real number then the binary operation* div *is defined as follows:*

$$x \text{ div } y = \text{int}\left(\frac{x}{y}\right)$$

The following table gives some examples of this operation:

$$\begin{array}{lcl}
4.3 \text{ div } 3 & = & 1 \\
4.3 \text{ div } -3 & = & -2 \\
-4.3 \text{ div } 3 & = & -2 \\
-4.3 \text{ div } -3 & = & 1 \\
4 \text{ div } 3 & = & 1 \\
4 \text{ div } -3 & = & -2 \\
-4 \text{ div } 3 & = & -2 \\
-4 \text{ div } -3 & = & 1
\end{array}$$

Some use is also made of the function abs which is defined as follows:

Definition 60 (abs) *If* x *is a real number then*

$$\text{abs}(x) = \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \text{if} \quad x < 0 \end{cases}$$

This function returns the 'absolute value' of a real number.

1.2 Representing pitch systems and pitch in *MIPS*

This section is devoted to describing how pitch systems and pitch are represented in *MIPS*.

1.2.1 The concept of a *MIPS* pitch system

The intuitive concept of an equal-tempered pitch system is modelled in *MIPS* by a mathematical concept called a *pitch system*. A *MIPS* pitch system is defined as follows:

Definition 61 (Pitch system) *An object* ψ *is a well-formed* pitch system *if and only if it is an ordered quadruple*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

such that the following conditions are satisfied:

1. μ_{c} *is a natural number called the* chromatic modulus*;*
2. μ_{m} *is a natural number called the* morphetic modulus*;*
3. $\mu_{\mathrm{c}} \geq \mu_{\mathrm{m}}$*;*
4. f_0 *is a positive real number called the* standard frequency*;*
5. $p_{\mathrm{c},0}$ *is an integer called the* standard chromatic pitch*.*

The symbols used to represent *MIPS* concepts will be used consistently throughout this document so the reader is advised to memorize each symbol as it is introduced.

The chromatic modulus μ_{c} of a pitch system indicates the number of equal intervals into which the octave is divided. For example, for the Western 12-tone equal-tempered system, the chromatic modulus is 12. The concept of chromatic modulus is essentially identical to the concept of *chromatic cardinality* defined by Clough and Douthett ([CD91, 94]). It also corresponds to the value N in Cambouropoulos' 'N-tone discrete equal-tempered pitch space' ([Cam98, 50], [Cam96, 234]) and to the value that Agmon customarily labels a in his formal representation of the diatonic system ([Agm89, 11], [Agm96, 44]). In Balzano's exploration of the group-theoretic properties of 'equal-tempered systems of n-fold octave division' ([Bal80, 66]), the value n corresponds to the *MIPS* chromatic modulus.

The morphetic modulus is equal to the number of notes in scales within the pitch system. More precisely, it indicates the number of different functional categories that a pitch can have within a key within the pitch system. For example, for the Western tonal system, the morphetic modulus is 7 corresponding to the seven different letter-names (A to G) used in the Western pitch notation system.

The Western pitch notation system has evolved to use 7 different letter-names because, according to traditional tonal theory, each pitch in a piece of tonal music can be understood to have one of seven different tonal functions (tonic, supertonic, mediant,...) within the key that operates at the location in the music where the pitch occurs. Pitches with the same tonal function in the same key have the same letter-name. This relates to the idea that the pitch structure of Western tonal music can be interpreted using the traditional, 7-note, major and minor scales.

The concept of morphetic modulus is essentially identical to the concept of *diatonic cardinality* defined by Clough and Douthett ([CD91, 94]). It also corresponds to the value M in Cambouropoulos' 'M-tone scale' ([Cam98, 50–51], [Cam96, 234–235]). In Agmon's work, the value that corresponds to morphetic modulus is customarily denoted b ([Agm89, 11], [Agm96, 44]).

So, for example, if a musical style was based on anhemitonic pentatonic scales embedded in a 12-note chromatic, then its pitch system would have a morphetic modulus of 5 and a chromatic modulus of 12; and for a musical style based on the equipentatonic scale—a system that uses 5-note scales embedded in a 5-note chromatic—both the chromatic modulus and the morphetic modulus would be 5.

Thus, whereas the chromatic modulus tells us something about the *physical* structure of the pitch system (the number of equal frequency intervals into which an octave is divided), the morphetic modulus tells us something about the *cognitive* structure of the pitch system (the number of notes in the scales that are used in the pitch system).

1.2.2 The concept of a *MIPS* pitch

The concept of a *MIPS pitch* models the intuitive concept of a pitch within an equal-tempered pitch system and its associated system of notation. It is defined as follows:

Definition 62 (Pitch) *An object p is a well-formed* pitch *in a pitch system if and only if it is an ordered pair*

$$p = [p_\mathrm{c}, p_\mathrm{m}]$$

that satisfies the following conditions:

1. p_c *is an integer called the* chromatic pitch*;*

2. p_m *is an integer called the* morphetic pitch*.*

The chromatic pitch represents the frequency associated with the pitch in the equal-tempered system.[5] In fact, given a pitch system,

$$\psi = [\mu_\mathrm{c}, \mu_\mathrm{m}, f_0, p_{\mathrm{c},0}]$$

the frequency of a pitch in ψ can be calculated from its chromatic pitch using the standard frequency f_0 and the standard chromatic pitch $p_{\mathrm{c},0}$ (see Definition 66 on page 17 below). In the Western, 12-tone, equal-tempered system, the chromatic pitch associated with a note in a score can be thought of as indicating the key on a normal piano keyboard that must be pressed in order to play the note. A rise of one semitone results in an increase of 1 in chromatic pitch and a fall of one semitone results in a decrease of 1 in chromatic pitch. If one specifies that a chromatic pitch of 0 is associated with the lowest $A\natural$ on a normal piano keyboard ($A\natural_0$) then the chromatic pitch of $G\sharp_0$ is -1 and the chromatic pitch associated with middle C ($C\natural_4$) is 39.[6] Figure 1.5 shows a variety of notes in the Western 12-tone equal-tempered pitch system, each labelled with its *MIPS* pitch. The first element in each *MIPS* pitch indicates the chromatic pitch associated with the note.

In Western staff notation, the morphetic pitch of a note is determined by

1. the vertical position of the note-head on the staff,

2. the clef in operation on the staff at the location of the note, and

3. the transposition of the staff.

---

[5] See section 1.1.1 for a discussion of the relationship and distinction between pitch and frequency.

[6] Pitch names will be denoted throughout this document using the A.S.A. pitch naming system. In this system, the pitch of middle C is denoted $C\natural_4$, the C an octave above middle C is denoted $C\natural_5$. Multiple sharps and flats will be denoted with the appropriate number of $\sharp$s and $\flat$s. The double-sharp symbol will not be used. For example, $C\sharp\sharp_4$ has a sounding pitch two semitones above middle C. $C\sharp\sharp_4$ has the same sounding pitch within an equal-tempered system as $B\sharp\sharp\sharp_3$ and $D\natural_4$. The octave number of a pitch-name within the A.S.A system is always the same as that of the closest C below it *on the staff*. Thus the sounding pitch of $B\sharp_3$ within a 12-tone equal-tempered system is one semitone higher than that of $C\flat_4$. See section 1.4.1 for algorithms for converting between *MIPS* pitches and A.S.A. pitch names.

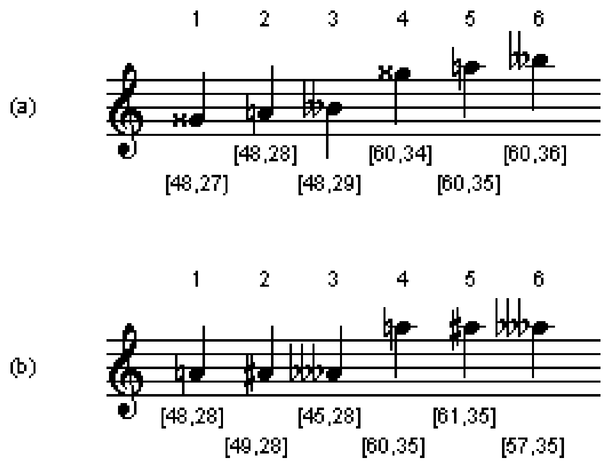


Figure 1.5: Examples of *MIPS* pitches in the Western staff notation system.

The morphetic pitch of a note is independent of the sounding pitch of the note and independent of its chromatic pitch. It indicates only the vertical position of the note on the staff. If the morphetic pitch of $A\natural_0$ is defined to be 0 then the morphetic pitch of $B\flat\flat_0$ is 1 and the morphetic pitch of $C\flat\flat\flat_1$ is 2 even though all three have the same sounding pitch in an equal-tempered system and would be performed by pressing the same key on a piano keyboard. The second element in each *MIPS* pitch in Figure 1.5 indicates the morphetic pitch of the note.

In Figure 1.5 (a) notes 1, 2 and 3 have the same chromatic pitch but different morphetic pitches and in Figure 1.5 (b) notes 1, 2 and 3 have the same morphetic pitch but different chromatic pitches. This illustrates the fact that morphetic pitch and chromatic pitch are mutually independent.

1.2.3 Calculating the chromatic pitch, morphetic pitch and frequency of a pitch

It is useful to define functions for calculating certain values from a *MIPS* pitch. The following two definitions provide functions for finding the chromatic pitch and morphetic pitch of a *MIPS* pitch:

Definition 63 (Chromatic pitch of a pitch) *If $p = [p_{\mathrm{c}}, p_{\mathrm{m}}]$ is a pitch in a well-formed pitch system then the following function returns the* chromatic pitch *of p:*

$$\mathrm{p_c}\left(p\right) = p_{\mathrm{c}}$$

Definition 64 (Morphetic pitch of a pitch) *If $p = [p_{\mathrm{c}}, p_{\mathrm{m}}]$ is a pitch in a well-formed pitch system then the following function returns the* morphetic pitch *of p:*

$$\mathrm{p_m}\left(p\right) = p_{\mathrm{m}}$$

These two definitions can be used to prove the following simple but useful theorem:

Theorem 65 *If ψ is a pitch system and p is a pitch in ψ then*

$$p = \left[\mathrm{p_c}\left(p\right), \mathrm{p_m}\left(p\right)\right]$$

(The reader is reminded that the proof of each theorem stated in the main body of the document is given in Chapter 4.)

The following definition provides a function for returning the frequency of a pitch within a *MIPS* pitch system[7]:

Definition 66 (Frequency of a pitch) *If p is a pitch in the pitch system*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

then the function

$$\mathrm{f}(p) = f_0 \times 2^{(\mathrm{p_c}(p) - p_{\mathrm{c},0})/\mu_{\mathrm{c}}}$$

returns the frequency *of p.*

This function assumes that the pitch system being modelled is an equal-tempered pitch system in which each octave is divided into μ_{c} *equal* intervals. To model a non-equal-tempered pitch system in *MIPS*, this function would have to be modified appropriately. In principle, if the frequency of a pitch within a pitch system can be calculated from its *MIPS* pitch, then the pitch system can be modelled in *MIPS* (provided that one defines an appropriate frequency function in place of that given in Definition 66).

Enough concepts have now been introduced for a number of concrete examples of *MIPS* pitch systems to be presented.

1.2.4 Some examples of *MIPS* pitch systems

A *MIPS* pitch system,

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

models a pitch system that employs scales containing μ_{m} notes, performed in an equal-tempered tuning system where the frequency f_0 is associated with the chromatic pitch $p_{\mathrm{c},0}$ and where the octave is divided into μ_{c} equal frequency intervals.

In the 12-tone equal-tempered system commonly used in the West, the frequency of the pitch $A\natural_4$ is commonly set to 440Hz. If $A\natural_0$ is defined to have a *MIPS* pitch of $[0,0]$ then the Western tonal equal-tempered pitch system and its associated staff-notation system which is designed to represent music constructed using 7-note scales, would be represented in *MIPS* as follows:

$$\psi_{\mathrm{W}} = [12, 7, 440, 48] \tag{1.1}$$

Within this pitch system, the pitch of $C\natural_4$ (middle C) is $[39,23]$. Therefore, using the frequency function defined above (Definition 66), the frequency of $C\natural_4$ is given by

$$\begin{aligned}\mathrm{f}([39,23]) &= 440 \times 2^{(39-48)/12} \\ &\approx 262\mathrm{Hz}\end{aligned}$$

As another example, consider the *MIPS* pitch system

$$\psi_{\mathrm{AP}} = [12, 5, 440, 48] \tag{1.2}$$

This models a pitch system that employs 5-note scales, embedded in a 12-tone equal-tempered chromatic, tuned in the same way as that used in ψ_{W} (see Equation 1.1). An example of such a system would be one that uses anhemitonic pentatonic scales (hence the 'AP' suffix on ψ_{AP}).

Just as the Western equal-tempered system divides the octave into 12 equal intervals, each of 100 cents, so the 'equipentatonic' system divides the octave into 5 equal intervals each of 240 cents. An equipentatonic

[7]See section 1.1.1 for a discussion of the relationship and distinction between pitch and frequency

system in which the pitch $[0,0]$ has the same frequency as $A\natural_0$ in the Western system modelled by ψ_{W} would be represented in *MIPS* as follows:

$$\psi_{\mathrm{EP}} = [5, 5, 440, 20] \tag{1.3}$$

As a final example, according to Clough *et al.* ([CDRR93, 36]) the classical Indian pitch system is supposed to have consisted of a ' "chromatic" universe of 22 microtonal divisions of the octave (the *śrutis*)' in which scales containing seven degrees or 'svaras' were constructed. This system was almost definitely not strictly equal-tempered but by appropriately changing the function defined in Definition 66, one could model this classical Indian pitch system in *MIPS* using a pitch system such as

$$\psi_{\mathrm{I}} = [22, 7, 440, 88] \tag{1.4}$$

(Again, in this pitch system, the value of $p_{\mathrm{c},0}$ is chosen (arbitrarily) so that the pitch $[0,0]$ has the same sounding pitch as $A\natural_0$ in the Western tonal system.)

1.2.5 Analogues of pitch, chromatic pitch and morphetic pitch in other pitch representation systems

The pitch representation system devised by Brinkman ([Bri90, 119–135]) is designed to represent the Western tonal pitch system and its associated staff notation system. Brinkman does not explicitly generalise his system to all equal-tempered pitch systems. The *MIPS* pitch system that corresponds to the one modelled by Brinkman is

$$\psi_{\mathrm{Brinkman}} = [12, 7, 440, 57] \tag{1.5}$$

where the pitch-name $C\natural_0$ is assigned a *MIPS* pitch of $[0,0]$. The chromatic pitch of a pitch in ψ_{Brinkman} corresponds to Brinkman's *continuous pitch code* (abbreviated *cpc*) ([Bri90, 122]) and a morphetic pitch in ψ_{Brinkman} corresponds to Brinkman's *continuous name code* (*cnc*) ([Bri90, 126]). Brinkman's *continuous binomial representation* (*cbr*) ([Bri90, 133]) is essentially identical to a *MIPS* pitch in ψ_{Brinkman}.

Unlike Brinkman, Agmon explicitly generalises his pitch representation system to any equal-tempered system. In Agmon's system, the function of a *MIPS* pitch is served by the integer pair that he consistently labels (x, y), the value x corresponding to chromatic pitch and the value y corresponding to morphetic pitch ([Agm96, 44], [Agm89, 11]).

MIDI note numbers ([Rot92, 25, 143, 214], [MMA96, 10]) are similar to chromatic pitches in *MIPS*. However, whereas a chromatic pitch can take any integer value whatsoever, a MIDI note number must be an integer greater than or equal to 0 and less than 128. The frequency of the pitch associated with a MIDI note number depends on the note mapping and tuning of the instrument producing the tone ([Rot92, 143]). However, it is common for a MIDI note number of 60 to correspond to $C\natural_4$, and in this particular case, the MIDI note numbers are identical to a subset of the values that can be taken by a chromatic pitch in the pitch system

$$\psi_{\mathrm{MIDI}} = [12, 7, 440, 69] \tag{1.6}$$

There is no analogue of morphetic pitch in the MIDI system and therefore nothing that corresponds to the *MIPS* concept of a pitch.

1.2.6 Chromatic pitch equivalence, chroma and chroma equivalence

Figure 1.6 shows a number of notes which the reader should interpret as being in the normal Western 12-tone equal-tempered system (i.e. ψ_{W}—see Equation 1.1 above). The pitches of notes 1, 2 and 3 in Figure 1.6 are enharmonically equivalent. The pitches of notes 4, 5 and 6 in Figure 1.6 are also enharmonically equivalent.

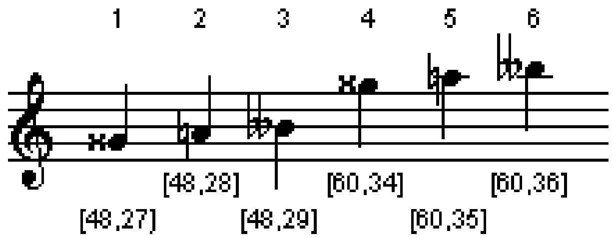


Figure 1.6: Examples of chromatic pitch equivalence and chroma equivalence in ψ_{W}.

The *MIPS* pitch of each note in ψ_{W} is given underneath the staff. Notes 1, 2 and 3 all have a chromatic pitch of 48 and notes 4, 5 and 6 all have a chromatic pitch of 60. In *MIPS*, two pitches have the same chromatic pitch if and only if they are enharmonically equivalent. The concept of enharmonic equivalence is therefore modelled in *MIPS* by the concept of *chromatic pitch equivalence* which is defined as follows:

Definition 125 (Chromatic pitch equivalence of pitches) *Two pitches p_1 and p_2 in a well-formed pitch system are* chromatic pitch equivalent *if and only if*

$$\mathrm{p_c}(p_1) = \mathrm{p_c}(p_2)$$

The fact that two pitches are chromatic pitch equivalent will be denoted

$$p_1 \equiv_{\mathrm{p_c}} p_2$$

All six pitches in Figure 1.6 are also 'sounding octave equivalent' in the sense that the frequency of the sounding pitch of notes 1, 2 and 3 would be 1/2 of the frequency of the sounding pitch of notes 4, 5 and 6 in an equal-tempered system. In *MIPS*, two notes are 'sounding octave equivalent' in this sense if and only if they have the same *chroma*. The chroma of a *MIPS* pitch is defined as follows:

Definition 71 (Chroma of a pitch) *If p is a pitch in a pitch system*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

then the following function returns the chroma *of p:*

$$\mathrm{c}(p) = \mathrm{p_c}(p) \bmod \mu_{\mathrm{c}}$$

The concept of 'sounding octave equivalence' exhibited by the six notes in Figure 1.6 can be modelled in *MIPS* by the concept of *chroma equivalence* which is defined as follows:

Definition 130 (Chroma equivalence of pitches) *Two pitches p_1 and p_2 in a well-formed pitch system are* chroma equivalent *if and only if*

$$\mathrm{c}(p_1) = \mathrm{c}(p_2)$$

The fact that two pitches are chroma equivalent will be denoted

$$p_1 \equiv_{\mathrm{c}} p_2$$

The *MIPS* concept of a chroma is essentially identical to the concept of pitch class used by Babbitt ([Bab60]), Forte ([For73]), Rahn ([Rah80]), Morris ([Mor87]) and many other theorists concerned with the structure of atonal and 12-tone music. The term *chroma* has been used by researchers in the field of music cognition

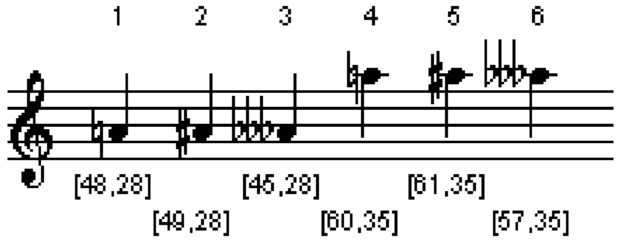


Figure 1.7: Examples of morphetic pitch equivalence and morph equivalence in ψ_{W}.

and perception for at least half a century to signify that quality of the pitch of a tone that makes it similar to the pitches of tones separated from it by one or more octaves. This perceptual similarity between the pitches of tones separated by one or more octaves has led cognitive psychologists to model musical pitch using a bidimensional model in which one dimension represents 'pitch level' or *tone height* and the other dimension—*tone chroma*—represents the position of a tone within its octave ([Deu82a, 272], [She82, 352], [WB82, 432–433]). Bachem used the term in this sense in 1950 ([Bac50]) and many other authors have used it since including Shepard ([She64], [She65], [She82]), Burns and Ward ([BW82, 246, 262–264], [WB82, 432–433]), Deutsch ([Deu82a, 272]), Dowling ([Dow91, 35]), and Cross, West and Howell ([CWH91, 212, 223–224]).

Brinkman's concept of pitch class (or *pc*) ([Bri90, 119–122]) is essentially identical to chroma in the *MIPS* pitch system ψ_{Brinkman} defined in Equation 1.5 above. Cambouropoulos also uses the term pitch class in this sense ([Cam98, 50], [Cam96, 234]) but unlike Brinkman, Cambouropoulos explicitly generalises the concept to any equal-tempered pitch system of 'N-tone' division that uses 'M-tone' scales. The *MIPS* concept of chroma is also essentially identical to the variable that Agmon consistently labels s in his definition of 'octave equivalence' ([Agm89, 11], [Agm96, 44]).

1.2.7 Morphetic pitch equivalence, morph and morph equivalence

The A.S.A. pitch names of notes 1, 2 and 3 in Figure 1.7 are, respectively $A\natural_4$, $A\sharp_4$ and $A\flat\flat\flat_4$.[8] All three notes have the same letter-name (A) and the same A.S.A. octave number (4) and this is represented in *MIPS* by the fact that they all have the same morphetic pitch (in this case, 28). This form of equivalence is therefore modelled in *MIPS* by the concept of *morphetic pitch equivalence* which is formally defined as follows:

Definition 126 (Morphetic pitch equivalence of pitches) *Two pitches p_1 and p_2 in a well-formed pitch system are* morphetic pitch equivalent *if and only if*

$$\mathrm{p_m}(p_1) = \mathrm{p_m}(p_2)$$

The fact that two pitches are morphetic pitch equivalent will be denoted

$$p_1 \equiv_{\mathrm{p_m}} p_2$$

Notes 4, 5 and 6 in Figure 1.7 are also morphetic pitch equivalent but notes 1 and 4 are not because their A.S.A. octave numbers are different. Nonetheless, all six notes in Figure 1.7 have the same letter-name (A) and this is represented in *MIPS* by the fact that they all have the same *morph*.[9] The *morph* of a *MIPS* pitch is defined as follows:

[8]See footnote 6 for an explanation of the logic behind A.S.A. pitch names.

[9]The name *morph* derives from the Greek word for 'shape' on an analogy with the derivation of the word *chroma* from the Greek word for 'colour'. If one property of a pitch is called its 'colour' then another one might as well be called its 'shape'!

Definition 76 (Morph of a pitch) *If p is a pitch in the pitch system*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

then the following function returns the morph *of p:*

$$\mathrm{m}\,(p) = \mathrm{p_m}\,(p) \bmod \mu_{\mathrm{m}}$$

The 'letter-name equivalence' exhibited by the six notes in Figure 1.7 is modelled in *MIPS* by the concept of *morph equivalence* which is formally defined as follows:

Definition 131 (Morph equivalence of pitches) *Two pitches p_1 and p_2 in a well-formed pitch system are* morph equivalent *if and only if*

$$\mathrm{m}\,(p_1) = \mathrm{m}\,(p_2)$$

The fact that two pitches are morph equivalent will be denoted

$$p_1 \equiv_{\mathrm{m}} p_2$$

Brinkman's concept of 'name class' (*nc*) ([Bri90, 124–126]) is essentially identical to morph within the *MIPS* pitch system ψ_{Brinkman} (see Equation 1.5). However Brinkman does not explicitly generalise his concept of 'name class' to other pitch systems. Cambouropoulos also uses the term 'name class' to refer to the concept in his *GPIR* that corresponds to morph in *MIPS*. In Agmon's definition of 'octave equivalence' ([Agm89, 11], [Agm96, 44]) the function that morph serves within *MIPS* is carried out by the variable that he consistently labels t.

In [Clo79], Clough elaborates a 'theory of diatonic pc sets' that corresponds to the morph set theory for a *MIPS* pitch system in which $\mu_{\mathrm{m}} = 7$ and the letter-name C in the Western diatonic system is represented by the morph 0. In [Clo80], Clough continues to use the term 'pitch class' for the concept that is called morph in *MIPS* but specifies that although 'the term *pitch class* (PC) will be employed in the usual sense', 'a universe of *seven* PC's is posited' ([Clo80, 468]). In [CD91], Clough and Douthett avoid using a concept that corresponds to morph in *MIPS* by considering 'subset[s] of d pcs selected from the chromatic universe of c pcs' which they label in the following way

$$D_{c,d} = \{D_0, D_1, D_2, \ldots, D_{d-1}\}$$

In this system, each D_k is a pitch class in the 12-tone chromatic (that is, D_k is a *chroma*) and the subscript k actually fulfills the function of morph since it indicates which chroma corresponds to which morph.

1.2.8 Chromatic octave and morphetic octave

If the notes in Figure 1.8 are interpreted as being in the equal-tempered pitch system ψ_{W}, then the frequency (and chromatic pitch) of note 1 ($B\sharp_4$) is higher than that of note 2 ($C\flat_5$). However, the A.S.A. octave number and morphetic pitch of note 1 is lower than that of note 2. This suggests the utility of distinguishing between two types of octave designation—one for sounding pitch (chromatic pitch) and one for morphetic pitch.

In *MIPS*, the *chromatic octave* of a pitch is defined as follows:

Definition 68 (Chromatic octave of a pitch) *If p is a pitch in the pitch system*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

then the following function returns the chromatic octave *of p:*

$$\mathrm{o_c}\,(p) = \mathrm{p_c}\,(p) \operatorname{div} \mu_{\mathrm{c}}$$

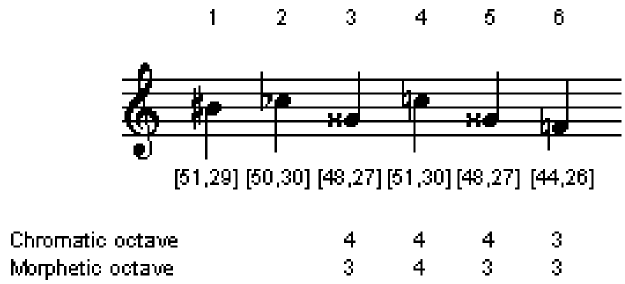


Figure 1.8: Examples of morphetic octave equivalence and chromatic octave equivalence in ψ_{W}.

The *morphetic octave* of a pitch is defined as follows:

Definition 69 (Morphetic octave of a pitch) *If p is a pitch in the pitch system*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

then the following function returns the morphetic octave *of p:*

$$\mathrm{o_m}(p) = \mathrm{p_m}(p) \text{ div } \mu_{\mathrm{m}}$$

In Figure 1.8, notes 3 and 4 have the same chromatic octave but different morphetic octaves; and notes 5 and 6 have the same morphetic octave but different chromatic octaves. This suggests the utility of defining two more equivalence relations: *morphetic octave equivalence* and *chromatic octave equivalence*. These are defined as follows:

Definition 128 (Chromatic octave equivalence of pitches) *Two pitches p_1 and p_2 in a well-formed pitch system are* chromatic octave equivalent *if and only if*

$$\mathrm{o_c}(p_1) = \mathrm{o_c}(p_2)$$

The fact that two pitches are chromatic octave equivalent will be denoted

$$p_1 \equiv_{\mathrm{o_c}} p_2$$

Definition 129 (Morphetic octave equivalence of pitches) *Two pitches p_1 and p_2 in a well-formed pitch system are* morphetic octave equivalent *if and only if*

$$\mathrm{o_m}(p_1) = \mathrm{o_m}(p_2)$$

The fact that two pitches are morphetic octave equivalent will be denoted

$$p_1 \equiv_{\mathrm{o_m}} p_2$$

We can now say, therefore, that in Figure 1.8, notes 3 and 4 are chromatic octave equivalent but not morphetic octave equivalent; and that notes 5 and 6 are morphetic octave equivalent but not chromatic octave equivalent.

If one takes the *MIPS* pitch system ψ_{Brinkman} defined in Equation 1.5 and sets the pitch-name $C\natural_0$ to correspond to the *MIPS* pitch $[0, 0]$ then, for any pitch p in this pitch system, the morphetic octave is equal to the A.S.A. octave number. In other words, the octave number in the A.S.A. pitch naming system corresponds to morphetic octave in the *MIPS* pitch system ψ_{Brinkman} with the pitch name $C\natural 0$ set to correspond to the *MIPS* pitch $[0,0]$. As already mentioned above (see section 1.2.5), Brinkman's concept of 'continuous pitch code' corresponds to chromatic pitch within ψ_{Brinkman} and it can be shown that for any pitch p in any *MIPS* pitch system

$$\psi = [\mu_{\text{c}}, \mu_{\text{m}}, f_0, p_{\text{c},0}]$$

it is true that

$$\text{p}_{\text{c}}(p) = (\text{o}_{\text{c}}(p) \times \mu_{\text{c}}) + \text{c}(p) \tag{1.7}$$

(See Theorem 75 in Chapter 4.) However, Brinkman states that his continuous pitch code, 'cpc', can be calculated using the following formula

$$cpc = (oct \times 12) + pc \tag{1.8}$$

where *oct* is the A.S.A. octave number and *pc* is his 'pitch class' which corresponds exactly to chroma in ψ_{Brinkman}. But, as mentioned above, A.S.A octave number corresponds exactly to morphetic octave in the pitch system ψ_{Brinkman} when $C\natural 0$ is set to correspond to the *MIPS* pitch $[0,0]$. Therefore, in *MIPS* terms, Brinkman's definition of 'cpc' can be stated as follows:

$$\text{p}_{\text{c}}(p) = (\text{o}_{\text{m}}(p) \times \mu_{\text{c}}) + \text{c}(p) \tag{1.9}$$

where $\mu_{\text{c}} = 12$ and the pitch $[0,0]$ corresponds to $C\natural 0$. But Equation 1.9 and Equation 1.7 together imply that

$$\text{o}_{\text{m}}(p) = \text{o}_{\text{c}}(p)$$

which was shown above not to be true in general (see, for example, note 3 in Figure 1.8). This, in turn, implies that at least one of Equation 1.9 and Equation 1.7 is incorrect. Since 1.7 can be shown to be true, this implies that 1.9 is incorrect.

An example will serve to demonstrate that Equation 1.9 is incorrect. Let $p_1 = [48, 27]$, the *MIPS* pitch representation of $B\sharp 3$ in ψ_{Brinkman} with $C\natural 0$ corresponding to $[0,0]$. From Definition 71 it follows that

$$\begin{aligned} \text{c}(p_1) &= \text{p}_{\text{c}}(p_1) \bmod \mu_{\text{c}} \\ &= 48 \bmod 12 \\ &= 0 \end{aligned} \tag{1.10}$$

and from Definition 69 it follows that

$$\begin{aligned} \text{o}_{\text{m}}(p_1) &= \text{p}_{\text{m}}(p_1) \operatorname{div} \mu_{\text{m}} \\ &= 27 \operatorname{div} 7 \\ &= 3 \end{aligned} \tag{1.11}$$

Substituting into Equation 1.9 the values of $\text{o}_{\text{m}}(p_1)$ and $\text{c}(p_1)$ found in Equations 1.10 and 1.11 gives

$$\begin{aligned} \text{p}_{\text{c}}(p) &= (\text{o}_{\text{m}}(p) \times \mu_{\text{c}}) + \text{c}(p) \\ &= (3 \times 12) + 0 \\ &= 36 \end{aligned} \tag{1.12}$$

which we know to be incorrect because p_1 was defined to be equal to $[48, 27]$. In fact, Equation 1.12 implies that $B\sharp 3$ has the same frequency as $C\natural 3$ which is clearly incorrect. This arises because $\text{o}_{\text{c}}(p_1) \neq \text{o}_{\text{m}}(p_1)$. Equations 1.10 and 1.11 are known to be correct therefore Equation 1.9 is incorrect.

It is interesting to note that in his definition of 'continuous binomial representation' ('cbr') ([Bri90, 133–134]) (which corresponds to pitch in the *MIPS* pitch system ψ_{Brinkman}), Brinkman correctly specifies that

$$[cpc, cnc] = [(poct \times 12) + pc, (noct \times 7) + nc]$$

where *poct* corresponds to chromatic octave in ψ_{Brinkman} and *noct* corresponds to morphetic octave in the same pitch system with $C\natural_0$ represented by $[0, 0]$. However, Brinkman claims that one only needs to use 'separate octave designators' if one needs 'to represent notes with any number of accidentals' and goes on to claim that 'in practice this is not really necessary, so long as we are willing to accept the limitation of quintuple accidentals and quintuple augmentation and diminution for intervals'. As shown in the previous paragraph, this is not true: one needs to distinguish between chromatic and morphetic octave whenever 'the notated pitch (cnc) is in a different octave from the sounding pitch (cpc)' ([Bri90, 134]) and this occurs even for pitches such as $C\flat_4$ or $B\sharp_3$ which have just a single sharp or flat.

It is therefore disappointing that Brinkman downplays the importance of distinguishing between chromatic and morphetic octave and, as a consequence, incorrectly concludes that 'we can use a single octave number, that in which the pitch is notated, and calculate the correct pitch level with minimal computation' ([Bri90, 134]).

Like Brinkman, Cambouropoulos decides to use only morphetic octave in his *GPIR*. However this, in itself, does not cause a problem because he explicitly represents the accidental of the pitch name. In Cambouropoulos' *GPIR*, a pitch is represented as an ordered quadruple, $[nc, mdf, pc, oct]$, where *nc* and *pc* are name class and pitch class as in Brinkman's system, *oct* is essentially the same as morphetic octave and *mdf* is a numerical representation of the accidental with -1 corresponding to $\flat$, 0 corresponding to $\natural$, 1 corresponding to $\sharp$ and so on. Cambouropoulos specifies that *mdf* takes values from $\{-u, \ldots, -1, 0, 1, \ldots, u\}$ where 'u is the number of pitch interval units in the largest scale-step interval' ([Cam98, 50]). This implies that Cambouropoulos' system cannot be used to represent notes with more than two sharps or flats. The reason for this restriction is unclear.

1.2.9 The concept of a *MIPS* pitch interval

In *MIPS*, the traditional concept of a pitch interval is modelled by the *MIPS* concept of a *pitch interval*. However, before defining the concept of a *MIPS* pitch interval, it is necessary to define the ideas of *morphetic pitch interval* and *chromatic pitch interval*:

Definition 236 (Chromatic pitch interval) *If* $\mathrm{p_{c,1}}$ *and* $\mathrm{p_{c,2}}$ *are two chromatic pitches in a well-formed pitch system* ψ*, then the* chromatic pitch interval from $\mathrm{p_{c,1}}$ to $\mathrm{p_{c,2}}$ *is defined and denoted as follows:*

$$\Delta\,\mathrm{p_c}\left(p_{\mathrm{c},1}, p_{\mathrm{c},2}\right) = p_{\mathrm{c},2} - p_{\mathrm{c},1}$$

Definition 240 (Morphetic pitch interval) *If* $\mathrm{p_{m,1}}$ *and* $\mathrm{p_{m,2}}$ *are two morphetic pitches in a well-formed pitch system* ψ*, then the* morphetic pitch interval from $\mathrm{p_{m,1}}$ to $\mathrm{p_{m,2}}$ *is defined and denoted as follows:*

$$\Delta\,\mathrm{p_m}\left(p_{\mathrm{m},1}, p_{\mathrm{m},2}\right) = p_{\mathrm{m},2} - p_{\mathrm{m},1}$$

It is now possible to present definitions for the chromatic pitch interval between two pitches and the morphetic pitch interval between two pitches:

Definition 259 (Definition of $\Delta\,\mathrm{p_c}\left(p_1, p_2\right)$) *If* p_1 *and* p_2 *are two pitches in a pitch system* ψ *then the* chromatic pitch interval from p_1 to p_2 *is defined and denoted as follows:*

$$\Delta\,\mathrm{p_c}\left(p_1, p_2\right) = \Delta\,\mathrm{p_c}\left(\mathrm{p_c}\left(p_1\right), \mathrm{p_c}\left(p_2\right)\right)$$

Definition 261 (Definition of $\Delta\,\mathrm{p_m}(p_1, p_2)$) *If p_1 and p_2 are two pitches in a pitch system ψ then the* morphetic pitch interval from p_1 to p_2 *is defined and denoted as follows:*

$$\Delta\,\mathrm{p_m}(p_1, p_2) = \Delta\,\mathrm{p_m}(\mathrm{p_m}(p_1), \mathrm{p_m}(p_2))$$

The concept of a *MIPS* pitch interval can then be defined as follows:

Definition 265 (Pitch interval) *If p_1 and p_2 are two pitches in a pitch system ψ then the* pitch interval from p_1 to p_2 *is defined and denoted as follows:*

$$\Delta\,\mathrm{p}(p_1, p_2) = [\Delta\,\mathrm{p_c}(p_1, p_2), \Delta\,\mathrm{p_m}(p_1, p_2)]$$

It is useful to define two functions, one for calculating the chromatic pitch interval of a pitch interval and one for calculating the morphetic pitch interval of a pitch interval:

Definition 266 (Chromatic pitch interval of a pitch interval) *If p_1 and p_2 are any two pitches in a pitch system ψ then*

$$\Delta p = \Delta\,\mathrm{p}(p_1, p_2) \Rightarrow \Delta\,\mathrm{p_c}(\Delta p) = \Delta\,\mathrm{p_c}(p_1, p_2)$$

Definition 268 (Morphetic pitch interval of a pitch interval) *If p_1 and p_2 are any two pitches in a pitch system ψ then*

$$\Delta p = \Delta\,\mathrm{p}(p_1, p_2) \Rightarrow \Delta\,\mathrm{p_m}(\Delta p) = \Delta\,\mathrm{p_m}(p_1, p_2)$$

These two definitions can be used to prove the following theorems which provide formulae for calculating the chromatic pitch interval of a pitch interval and the morphetic pitch interval of a pitch interval:

Theorem 269 (Formula for $\Delta\,\mathrm{p_m}(\Delta p)$) *If $\Delta p = [\Delta p_\mathrm{c}, \Delta p_\mathrm{m}]$ in a pitch system ψ then*

$$\Delta\,\mathrm{p_m}(\Delta p) = \Delta p_\mathrm{m}$$

Theorem 267 (Formula for $\Delta\,\mathrm{p_c}(\Delta p)$) *If $\Delta p = [\Delta p_\mathrm{c}, \Delta p_\mathrm{m}]$ in a pitch system ψ then*

$$\Delta\,\mathrm{p_c}(\Delta p) = \Delta p_\mathrm{c}$$

It is now possible to define a function for transposing a chromatic pitch by a chromatic pitch interval:

Definition 426 (Definition of $\tau_{\mathrm{p_c}}(p_\mathrm{c}, \Delta p_\mathrm{c})$) *If ψ is a pitch system and $p_{\mathrm{c},1}$ and $p_{\mathrm{c},2}$ are chromatic pitches in ψ and Δp_c is a chromatic pitch interval in ψ then*

$$\Delta p_\mathrm{c} = \Delta\,\mathrm{p_c}(p_{\mathrm{c},1}, p_{\mathrm{c},2}) \Rightarrow \tau_{\mathrm{p_c}}(p_{\mathrm{c},1}, \Delta p_\mathrm{c}) = p_{\mathrm{c},2}$$

This definition can be used in conjunction with other *MIPS* definitions and theorems to prove the following theorem which provides us with a formula for calculating the chromatic pitch that results when one transposes a chromatic pitch by a chromatic pitch interval:

Theorem 427 (Formula for $\tau_{\mathrm{p_c}}(p_\mathrm{c}, \Delta p_\mathrm{c})$) *If ψ is a pitch system and p_c is a chromatic pitch in ψ and Δp_c is a chromatic pitch interval in ψ then*

$$\tau_{\mathrm{p_c}}(p_\mathrm{c}, \Delta p_\mathrm{c}) = p_\mathrm{c} + \Delta p_\mathrm{c}$$

The definition of the morphetic pitch transposition function is strictly analogous to that of the chromatic pitch transposition function:

Definition 431 (Definition of $\tau_{\mathrm{p_m}}(p_{\mathrm{m}}, \Delta p_{\mathrm{m}})$) *If ψ is a pitch system and $p_{\mathrm{m},1}$ and $p_{\mathrm{m},2}$ are morphetic pitches in ψ and Δp_{m} is a morphetic pitch interval in ψ then*

$$\Delta p_{\mathrm{m}} = \Delta\,\mathrm{p_m}(p_{\mathrm{m},1}, p_{\mathrm{m},2}) \Rightarrow \tau_{\mathrm{p_m}}(p_{\mathrm{m},1}, \Delta p_{\mathrm{m}}) = p_{\mathrm{m},2}$$

This definition can be used in conjunction with other *MIPS* definitions and theorems to prove the following theorem which provides us with a formula for calculating the morphetic pitch that results when a morphetic pitch is transposed by a morphetic pitch interval:

Theorem 432 (Formula for $\tau_{\mathrm{p_m}}(p_{\mathrm{m}}, \Delta p_{\mathrm{m}})$) *If ψ is a pitch system and p_{m} is a morphetic pitch in ψ and Δp_{m} is a morphetic pitch interval in ψ then*

$$\tau_{\mathrm{p_m}}(p_{\mathrm{m}}, \Delta p_{\mathrm{m}}) = p_{\mathrm{m}} + \Delta p_{\mathrm{m}}$$

It is now possible to define the pitch transposition function:

Definition 441 (Definition of $\tau_{\mathrm{p}}(p, \Delta p)$) *If ψ is a pitch system and p_1 and p_2 are pitches in ψ and Δp is a pitch interval in ψ then*

$$\Delta p = \Delta\,\mathrm{p}(p_1, p_2) \Rightarrow \tau_{\mathrm{p}}(p_1, \Delta p) = p_2$$

This definition can be used in conjunction with certain other *MIPS* definitions and theorems to prove the following theorem which provides us with a formula for calculating the pitch that results when a *MIPS* pitch is transposed by a *MIPS* pitch interval:

Theorem 442 (Formula for $\tau_{\mathrm{p}}(p, \Delta p)$) *If ψ is a pitch system and p is a pitch in ψ and Δp is a pitch interval in ψ then*

$$\tau_{\mathrm{p}}(p, \Delta p) = \left[\tau_{\mathrm{p_c}}(\mathrm{p_c}(p), \Delta\,\mathrm{p_c}(\Delta p)), \tau_{\mathrm{p_m}}(\mathrm{p_m}(p), \Delta\,\mathrm{p_m}(\Delta p))\right]$$

The concept of the *inverse of a pitch interval* will now be be defined:

Definition 561 (Inverse of a pitch interval) *If ψ is a pitch system and Δp is a pitch interval in ψ and p is a pitch in ψ then the* inverse of *Δp, denoted $\iota_{\mathrm{p}}(\Delta p)$, is the pitch interval that satisfies the following equation*

$$\tau_{\mathrm{p}}(\tau_{\mathrm{p}}(p, \Delta p), \iota_{\mathrm{p}}(\Delta p)) = p$$

This definition together with other definitions and theorems from *MIPS* can be used to prove the following theorem which provides a formula for calculating the inverse of a pitch interval:

Theorem 563 *If*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

is a pitch system and Δp is a pitch interval in ψ then

$$\iota_{\mathrm{p}}(\Delta p) = \left[-\Delta\,\mathrm{p_c}(\Delta p), -\Delta\,\mathrm{p_m}(\Delta p)\right]$$

1.3 The genus representation of octave equivalence

This section is devoted to introducing, defining and discussing the genus representation of octave equivalence.

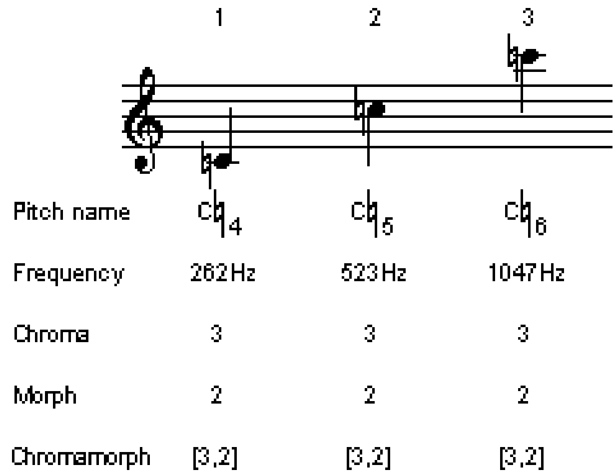


Figure 1.9: The traditional concept of 'octave equivalence' in ψ_{W}.

1.3.1 Chromamorph and genus

In traditional Western tonal theory, two notes are considered to be 'octave equivalent' if and only if they are an integer number of perfect octaves apart. Thus, in Figure 1.9, notes 1, 2 and 3 are 'octave equivalent' in this traditional sense. It is clear from Figure 1.9 that if two notes are separated by an integer number of perfect octaves then they will have the same chroma and the same morph. So as a first attempt at modelling the traditional concept of 'octave equivalence,' let us define the concept of a *chromamorph* and its associated equivalence relation, *chromamorph equivalence*:

Definition 80 (Chromamorph of a pitch) *If p is a pitch in a well-formed pitch system, then the following function returns the* chromamorph *of p:*

$$\mathrm{q}\left(p\right) = \left[\mathrm{c}\left(p\right), \mathrm{m}\left(p\right)\right]$$

Definition 132 (Chromamorph equivalence of pitches) *Two pitches p_1 and p_2 in a well-formed pitch system are* chromamorph equivalent *if and only if*

$$\mathrm{q}\left(p_1\right) = \mathrm{q}\left(p_2\right)$$

The fact that two pitches are chromamorph equivalent will be denoted

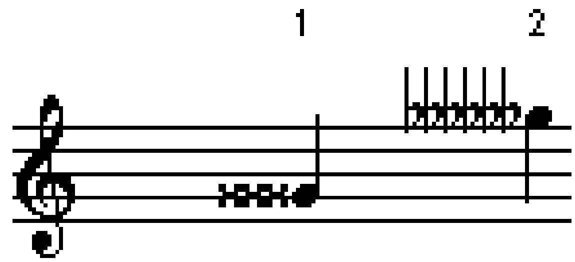
$$p_1 \equiv_{\mathrm{q}} p_2$$

Notes 1, 2 and 3 in Figure 1.9 all have the same chromamorph and are therefore chromamorph equivalent.

A number of authors have attempted to model the traditional concept of 'octave equivalence' using a concept essentially identical to chromamorph equivalence of pitches.[10] However, chromamorph equivalence does not correctly model the traditional concept of 'octave equivalence' within the 12-tone equal-tempered tonal pitch system and pitch notation system.

Notes 1 and 2 in Figure 1.10 have the same chromamorph—$[4, 6]$ in ψ_{W}. They are therefore chromamorph equivalent. However, the interval between them is certainly not an integer number of perfect octaves—it is,

[10] See, for example, Brinkman's 'binomial representation' ([Bri90, 128]) and Agmon's definition of 'octave equivalence' ([Agm89, 11], [Agm96, 44]).



	1	2
Chroma	4	4
Morph	6	6
Pitch	[52,27]	[52,34]
Chromamorph	[4,6]	[4,6]
Old genus	[6,6]	[6,-6]
Genus	[16,6]	[4,6]

Figure 1.10: The difference between genus and chromamorph.

in fact, a '12×diminished octave'. The two notes are therefore not 'octave equivalent' in the traditional tonal sense.

As defined above (Definition 71) the chroma of a pitch $p = [p_c, p_m]$ is given by the following equation:

$$\mathrm{c}\,(p) = p_c \bmod \mu_c$$

and the morph of $p = [p_c, p_m]$ (see Definition 76) is given by the following equation:

$$\mathrm{m}\,(p) = p_m \bmod \mu_m$$

Informally speaking, the chroma of a pitch is found by taking the chromatic pitch and subtracting the chromatic modulus a certain number of times until one has a remainder c that is between 0 and $\mu_c - 1$. The number of times we have to subtract the chromatic modulus from the chromatic pitch to get the chroma is equal to the chromatic octave (see Definition 68):

$$\mathrm{o_c}\,(p) = \mathrm{p_c}\,(p) \text{ div } \mu_c$$

Similarly, the morph of a pitch is found by taking the morphetic pitch and subtracting the morphetic modulus a certain number of times until one has a remainder m that is between 0 and $\mu_m - 1$. The number of times we have to subtract the morphetic modulus from the morphetic pitch to get the morph is equal to the morphetic octave (see Definition 69):

$$\mathrm{o_m}\,(p) = \mathrm{p_m}\,(p) \text{ div } \mu_m$$

But, of course, $\mathrm{o_m}\,(p)$ and $\mathrm{o_c}\,(p)$ for a given pitch are not necessarily the same because $\mathrm{p_c}\,(p)$ and $\mathrm{p_m}\,(p)$ are mutually independent and can each take any integer value.

For example, to find the chroma of note 1 in Figure 1.10 we find the least positive remainder when we divide the chromatic pitch (52) by the chromatic modulus. To do this in this case we effectively subtract the chromatic modulus from the chromatic pitch *four* times:

$$52 - (4 \times 12) = 4$$

To find the morph we find the least positive remainder when we divide the morphetic pitch by the morphetic modulus which, in this case involves subtracting the morphetic modulus *three* times from the morphetic pitch:

$$27 - (3 \times 7) = 6$$

To find the chroma of note 2 in Figure 1.10 we have to subtract the chromatic modulus *four* times from the chromatic pitch

$$52 - (4 \times 12) = 4$$

and to find the morph we subtract the morphetic modulus *four* times from the morphetic pitch

$$34 - (4 \times 7) = 6$$

For note 2, the chromatic octave is the same as the morphetic octave but for note 1, the chromatic octave is *not* equal to the morphetic octave. Let us define the concept of *octave difference* as follows:

Definition 81 (Octave difference of a pitch) *If p is a pitch in a well-formed pitch system, then the following function returns the* octave difference *of p:*

$$\mathrm{d_o}(p) = \mathrm{o_c}(p) - \mathrm{o_m}(p)$$

This implies that the octave difference of note 1 is

$$4 - 3 = 1$$

but the octave difference of note 2 is

$$4 - 4 = 0$$

For two notes to be 'octave equivalent' in the traditional tonal sense they must have not only the same morph and the same chroma *but also the same octave difference.*

This example suggests that we can achieve a correct representation of tonal octave equivalence simply by using a representation in which we replace the chroma in a chromamorph with a value that is the result of subtracting the chromatic modulus from the chromatic pitch the same number of times that we subtract the morphetic modulus from the morphetic pitch to get the morph. In *MIPS*, this replacement for the chroma in a chromamorph is called the *chromatic genus* of a pitch and it is defined as follows:

Definition 82 (Chromatic genus of a pitch) *If p is a pitch in a well-formed pitch system*

$$\psi = [\mu_\mathrm{c}, \mu_\mathrm{m}, f_0, p_{\mathrm{c},0}]$$

then the following function returns the chromatic genus *of p:*

$$\mathrm{g_c}(p) = \mathrm{p_c}(p) - \mu_\mathrm{c} \times \mathrm{o_m}(p)$$

This gives us a new representation of octave equivalence which in this document will be called *genus*. A genus is an ordered pair similar to a chromamorph, except that the first element is the chromatic genus of the pitch and the second element is the morph of the pitch. The genus of a pitch is defined as follows:

Definition 84 (Genus of a pitch) *If p is a pitch in a well-formed pitch system then the following function returns the* genus *of p:*

$$\mathrm{g}(p) = [\mathrm{g_c}(p), \mathrm{m}(p)]$$

The corresponding concept of *genus equivalence* is defined as follows:

Definition 135 (Genus equivalence of pitches) *Two pitches p_1 and p_2 in a well-formed pitch system are* genus equivalent *if and only if*

$$\mathrm{g}(p_1) = \mathrm{g}(p_2)$$

The fact that two pitches are genus equivalent will be denoted

$$p_1 \equiv_{\mathrm{g}} p_2$$

It can be shown (see Definition 87 in Chapter 4) that two pitches will have the same genus if and only if they have the same chroma, the same morph and the same octave difference.

Note that the genus of a pitch can be calculated directly from the chromatic pitch and morphetic pitch of the pitch. This implies that in order to find the genus of a pitch within a pitch system, one does not need first to know which sets within that pitch system correspond to the diatonic sets in the Western tonal system. Genus equivalence therefore correctly models the logic of the Western tonal pitch system and can be generalised to any other pitch system without first specifying which sets in that pitch system correspond to the diatonic sets of the Western tonal system.

1.3.2 Deriving *MIPS* objects from a genus

Given a *MIPS* pitch, it is possible to calculate its chromatic pitch (Definition 63), its morphetic pitch (Definition 64), its chroma (Definition 71) and so on. In a similar way, it is possible to calculate the chroma, morph, chromamorph and chromatic genus of a genus.

The function for returning the chromatic genus of a genus is defined as follows:

Definition 114 (Chromatic genus of a genus) *If g is the genus of a pitch p in a pitch system ψ then the function $\mathrm{g_c}(g)$ must return the chromatic genus of p. In other words, by definition, it must be true that*

$$(g = \mathrm{g}(p)) \Rightarrow (\mathrm{g_c}(g) = \mathrm{g_c}(p))$$

This definition can be used to prove the following theorem which provides a formula for calculating the chromatic genus of a genus:

Theorem 115 (Chromatic genus of a genus) *If $g = [g_\mathrm{c}, m]$ is the genus of a pitch in the pitch system ψ then*

$$\mathrm{g_c}(g) = g_\mathrm{c}$$

The function for returning the morph of a genus is defined as follows:

Definition 116 (Morph of a genus) *If g is the genus of a pitch p in a pitch system ψ then the function $\mathrm{m}(g)$ must return the morph of p. In other words, by definition, it must be true that*

$$(g = \mathrm{g}(p)) \Rightarrow (\mathrm{m}(g) = \mathrm{m}(p))$$

This definition can be used to prove the following theorem which provides a formula for calculating the morph of a genus:

Theorem 117 (Morph of a genus) *If $g = [g_\mathrm{c}, m]$ is the genus of a pitch in the pitch system ψ then*

$$\mathrm{m}(g) = m$$

Theorems 115 and 117 can be used to prove the following simple but useful theorem:

Theorem 118 *If g is a genus in a pitch system ψ then*

$$g = [\mathrm{g_c}(g), \mathrm{m}(g)]$$

The function for returning the chroma of a genus is defined as follows:

Definition 119 (Chroma of a genus) *If g is the genus of a pitch p in a pitch system ψ then the function $\mathrm{c}(g)$ must return the chroma of p. In other words, by definition, it must be true that*

$$(g = \mathrm{g}(p)) \Rightarrow (\mathrm{c}(g) = \mathrm{c}(p))$$

This definition can be used to prove the following theorem which provides a formula for calculating the chroma of a genus:

Theorem 120 (Chroma of a genus) *If g is the genus of a pitch in the pitch system*

$$\psi = [\mu_\mathrm{c}, \mu_\mathrm{m}, f_0, p_{\mathrm{c},0}]$$

then

$$\mathrm{c}(g) = \mathrm{g_c}(g) \bmod \mu_\mathrm{c}$$

Finally, the function that returns the chromamorph of a genus is defined as follows:

Definition 121 (Chromamorph of a genus) *If g is the genus of a pitch p in a pitch system ψ then the function $\mathrm{q}(g)$ must return the chromamorph of p. In other words, by definition, it must be true that*

$$(g = \mathrm{g}(p)) \Rightarrow (\mathrm{q}(g) = \mathrm{q}(p))$$

This definition can be used to prove the following theorem which provides a formula for calculating the chromamorph of a genus:

Theorem 122 (Chromamorph of a genus) *If g is the genus of a pitch in the pitch system ψ then*

$$\mathrm{q}(g) = [\mathrm{c}(g), \mathrm{m}(g)]$$

1.3.3 The concept of a genus interval

Before defining the concept of a *genus interval*, it is necessary to define that of a *morph interval*:

Definition 217 (Morph interval) *If m_1 and m_2 are two morphs in a well-formed pitch system*

$$\psi = [\mu_\mathrm{c}, \mu_\mathrm{m}, f_0, p_{\mathrm{c},0}]$$

then the morph interval from m_1 to m_2 *is given by the following equation:*

$$\Delta\,\mathrm{m}(m_1, m_2) = (m_2 - m_1) \bmod \mu_\mathrm{m}$$

This definition specifies how to calculate the morph interval from one morph to another. The following definition specifies how to calculate the morph interval from one *genus* to another.

Definition 228 (Morph interval between two genera) *If g_1 and g_2 are two genera in a pitch system ψ then the* morph interval from g_1 to g_2 *is defined and denoted as follows:*

$$\Delta\,\mathrm{m}(g_1, g_2) = \Delta\,\mathrm{m}(\mathrm{m}(g_1), \mathrm{m}(g_2))$$

The following definition provides a formula for calculating the *chromatic genus interval* between two genera:

Definition 230 (Chromatic genus interval between two genera) *If g_1 and g_2 are two genera in a pitch system*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

then the chromatic genus interval from g_1 to g_2 *is defined and denoted as follows:*

$$\Delta\,\mathrm{g_c}\,(g_1, g_2) = \mathrm{g_c}\,(g_2) - \mathrm{g_c}\,(g_1) - \mu_{\mathrm{c}} \times ((\mathrm{m}\,(g_2) - \mathrm{m}\,(g_1)) \operatorname{div} \mu_{\mathrm{m}})$$

The following definition uses Definitions 230 and 228 to provide an expression for the genus interval between two genera:

Definition 231 (Genus interval between two genera) *If g_1 and g_2 are two genera in a pitch system ψ then the* genus interval from g_1 to g_2 *is defined and denoted as follows:*

$$\Delta\,\mathrm{g}\,(g_1, g_2) = [\Delta\,\mathrm{g_c}\,(g_1, g_2)\,, \Delta\,\mathrm{m}\,(g_1, g_2)]$$

1.3.4 Transposing a genus

Having defined the concepts of genus and genus interval, it is now possible to define a function for transposing a genus by a genus interval:

Definition 421 (Genus transposition function) *If ψ is a pitch system and g_1 and g_2 are genera in ψ and Δg is a genus interval in ψ then the* genus transposition function *is defined as follows:*

$$\Delta\,\mathrm{g}\,(g_1, g_2) = \Delta g \Rightarrow \tau_{\mathrm{g}}\,(g_1, \Delta g) = g_2$$

This definition in combination with a number of other *MIPS* theorems and definitions can be used to prove a theorem which provides a formula for calculating the genus that results from transposing any given genus by any given genus interval. However, before stating this theorem, it is necessary to introduce three more concepts, namely, the *morph interval of a genus interval*, the *chromatic genus interval of a genus interval* and the *morph transposition function*.

The concept of the *morph interval of a genus interval* is defined as follows:

Definition 315 (Morph interval of a genus interval) *If g_1 and g_2 are two genera in a pitch system ψ then*

$$\Delta g = \Delta\,\mathrm{g}\,(g_1, g_2) \Rightarrow \Delta\,\mathrm{m}\,(\Delta g) = \Delta\,\mathrm{m}\,(g_1, g_2)$$

This definition can be used together with Definition 231 to prove the following theorem which provides a formula for calculating the morph interval of a genus interval:

Theorem 316 (Formula for morph interval of a genus interval) *If Δg is a genus interval in a pitch system ψ then*

$$\Delta g = [\Delta g_{\mathrm{c}}, \Delta m] \Rightarrow \Delta\,\mathrm{m}\,(\Delta g) = \Delta m$$

The concept of the *chromatic genus interval of a genus interval* is defined as follows:

Definition 309 (Chromatic genus interval of a genus interval) *If g_1 and g_2 are two genera in a pitch system ψ then*

$$\Delta g = \Delta\,\mathrm{g}\,(g_1, g_2) \Rightarrow \Delta\,\mathrm{g_c}\,(\Delta g) = \Delta\,\mathrm{g_c}\,(g_1, g_2)$$

This definition can be used together with Definition 231 to prove the following theorem which provides a formula for calculating the chromatic genus interval of a genus interval:

Theorem 310 (Formula for chromatic genus interval of a genus interval) *If Δg is a genus interval in a pitch system ψ then*

$$\Delta g = [\Delta g_{\mathrm{c}}, \Delta m] \Rightarrow \Delta\,\mathrm{g_c}\,(\Delta g) = \Delta g_{\mathrm{c}}$$

The *morph transposition function* is defined as follows:

Definition 411 (Morph transposition function) *If ψ is a pitch system and m_1 and m_2 are morphs in ψ and Δm is a morph interval in ψ then the* morph transposition function *is defined as follows:*

$$\Delta\,\mathrm{m}\,(m_1, m_2) = \Delta m \Rightarrow \tau_{\mathrm{m}}\,(m_1, \Delta m) = m_2$$

This definition, together with other theorems and definitions from *MIPS* can be used to prove the following theorem which provides a formula for calculating the morph that results when one transposes a morph by a morph interval:

Theorem 412 (Formula for morph transposition function) *If m is a morph and Δm is a morph interval in a pitch system*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

then

$$\tau_{\mathrm{m}}\,(m, \Delta m) = (m + \Delta m) \bmod \mu_{\mathrm{m}}$$

It is now possible to state a theorem that provides a formula for calculating the genus that results when one transposes a genus by a genus interval:

Theorem 422 (Formula for genus transposition function) *If*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

is a pitch system and g is a genus in ψ and Δg is a genus interval in ψ then

$$\tau_{\mathrm{g}}\,(g, \Delta g) = [\mathrm{g_c}\,(g) + \Delta\,\mathrm{g_c}\,(\Delta g) - \mu_{\mathrm{c}} \times ((\mathrm{m}\,(g) + \Delta\,\mathrm{m}\,(\Delta g)) \operatorname{div} \mu_{\mathrm{m}})\,, \tau_{\mathrm{m}}\,(\mathrm{m}\,(g)\,, \Delta\,\mathrm{m}\,(\Delta g))]$$

This theorem can be used in conjunction with a number of other *MIPS* definitions and theorems to prove the following two theorems that state certain important properties of the genus transposition function:

Theorem 424 *If ψ is a pitch system and g_1 and g_2 are genera in ψ and Δg is a genus interval in ψ then*

$$\tau_{\mathrm{g}}\,(g_1, \Delta g) = g_2 \iff \Delta\,\mathrm{g}\,(g_1, g_2) = \Delta g$$

Theorem 425 *If ψ is a pitch system and Δg_1 and Δg_2 are genus intervals in ψ and g is a genus in ψ then*

$$(\tau_{\mathrm{g}}\,(g, \Delta g_1) = \tau_{\mathrm{g}}\,(g, \Delta g_2)) \Rightarrow (\Delta g_1 = \Delta g_2)$$

1.3.5 Summation of genus intervals

The following definition provides a formula for calculating the sum of a collection of genus intervals:

Definition 491 (Summation of genus intervals) *If*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

is a pitch system and

$$\Delta g_1, \Delta g_2, \ldots \Delta g_n$$

is a collection of genus intervals in ψ *then*

$$\sigma_{\mathrm{g}}\left(\Delta g_1, \Delta g_2, \ldots \Delta g_n\right) = \left[\left(\sum_{k=1}^{n} \Delta\,\mathrm{g_c}\left(\Delta g_k\right)\right) - \mu_{\mathrm{c}} \times \left(\left(\sum_{k=1}^{n} \Delta\,\mathrm{m}\left(\Delta g_k\right)\right) \operatorname{div} \mu_{\mathrm{m}}\right), \left(\sum_{k=1}^{n} \Delta\,\mathrm{m}\left(\Delta g_k\right)\right) \bmod \mu_{\mathrm{m}}\right]$$

This definition in conjunction with other *MIPS* definitions and theorems can be used to prove the following theorem which provides a formula for calculating the genus that results when a genus is transposed by the sum of a collection of genus intervals:

Theorem 492 *If*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

is a pitch system, g is a genus in ψ *and*

$$\Delta g_1, \Delta g_2, \ldots \Delta g_n$$

is a collection of genus intervals in ψ *then*

$$\tau_{\mathrm{g}}\left(g, \sigma_{\mathrm{g}}\left(\Delta g_1, \Delta g_2, \ldots \Delta g_n\right)\right) = \left[\begin{array}{l} \mathrm{g_c}\left(g\right) + \left(\sum_{k=1}^{n} \Delta\,\mathrm{g_c}\left(\Delta g_k\right)\right) - \mu_{\mathrm{c}} \times \left(\left(\left(\sum_{k=1}^{n} \Delta\,\mathrm{m}\left(\Delta g_k\right)\right) + \mathrm{m}\left(g\right)\right) \operatorname{div} \mu_{\mathrm{m}}\right), \\ \\ \left(\mathrm{m}\left(g\right) + \left(\sum_{k=1}^{n} \Delta\,\mathrm{m}\left(\Delta g_k\right)\right)\right) \bmod \mu_{\mathrm{m}} \end{array}\right]$$

The following theorem simply states that transposing a genus g by the sum of a collection of genus intervals $\Delta g_1, \Delta g_2, \ldots \Delta g_n$ gives the same result as transposing g by Δg_1, then transposing the result of this transposition by Δg_2, the result of that transposition by Δg_3 and so on:

Theorem 493 *If* ψ *is a pitch system and*

$$\Delta g_1, \Delta g_2, \ldots \Delta g_n$$

is a collection of genus intervals in ψ *and g is a genus in* ψ *then*

$$\tau_{\mathrm{g}}\left(g, \sigma_{\mathrm{g}}\left(\Delta g_1, \Delta g_2, \ldots \Delta g_n\right)\right) = \tau_{\mathrm{g}}\left(\ldots \tau_{\mathrm{g}}\left(\tau_{\mathrm{g}}\left(g, \Delta g_1\right), \Delta g_2\right) \ldots, \Delta g_n\right)$$

1.3.6 Inverse of a genus interval

The *Inverse of a genus interval* is defined as follows:

Definition 494 (Inverse of a genus interval) *If* ψ *is a pitch system and* Δg *is a genus interval in* ψ *and g is a genus in* ψ *then the* inverse of Δg, *denoted* $\iota_{\mathrm{g}}\left(\Delta g\right)$, *is the genus interval that satisfies the following equation*

$$\tau_{\mathrm{g}}\left(\tau_{\mathrm{g}}\left(g, \Delta g\right), \iota_{\mathrm{g}}\left(\Delta g\right)\right) = g$$

The following theorem provides a formula for calculating the inverse of a genus interval:

Theorem 496 *If*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

is a pitch system and Δg *is a genus interval in* ψ *then*

$$\iota_{\mathrm{g}}\left(\Delta g\right) = \left[\mu_{\mathrm{c}} - \Delta\,\mathrm{g_c}\left(\Delta g\right), \left(-\Delta\,\mathrm{m}\left(\Delta g\right)\right) \bmod \mu_{\mathrm{m}}\right]$$

1.3.7 Exponentiation of a genus interval

The concept of *genus interval exponentiation* is defined as follows:

Definition 500 (Exponentiation of a genus interval) *Given that:*

1. ψ *is a pitch system;*
2. g *is a genus in* ψ*;*
3. Δg *is a genus interval in* ψ*;*
4. n *is an integer;*
5. k *is an integer and* $1 \leq k \leq \operatorname{abs}(n)$*;*
6. $\Delta g_{1,k} = \Delta g$ *for all* k*; and*
7. $\Delta g_{2,k} = \iota_{\mathrm{g}}(\Delta g)$ *for all* k*;*

then $\epsilon_{\mathrm{g},n}(\Delta g)$ *returns a genus interval that satisfies the following equation:*

$$\tau_{\mathrm{g}}(g, \epsilon_{\mathrm{g},n}(\Delta g)) = \begin{cases} \tau_{\mathrm{g}}(g, \sigma_{\mathrm{g}}(\Delta g_{1,1}, \Delta g_{1,2}, \ldots \Delta g_{1,n})) & \text{if} \quad n > 0 \\ g & \text{if} \quad n = 0 \\ \tau_{\mathrm{g}}(g, \sigma_{\mathrm{g}}(\Delta g_{2,1}, \Delta g_{2,2}, \ldots \Delta g_{2,-n})) & \text{if} \quad n < 0 \end{cases}$$

This definition effectively states that if n is a positive integer, then transposing a genus g by the nth power of the genus interval Δg must give the same result as that obtained when one transposes g by the sum of n genus intervals all of which are equal to Δg. The definition also states that if n is a negative integer, then the result of transposing a genus by the nth power of Δg must be the same as that obtained when one transposes g by the sum of a collection of $-n$ intervals, all of which are equal to the inverse of Δg. Transposing a genus by the zeroth power of any genus interval must result in no change in the genus.

The following theorem provides a formula for calculating the nth power of a genus interval:

Theorem 501 (Formula for $\epsilon_{\mathrm{g},n}(\Delta g)$) *If*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

is a pitch system and Δg *is a genus interval in* ψ *and* n *is an integer then*

$$\epsilon_{\mathrm{g},n}(\Delta g) = \begin{bmatrix} n \times \Delta\,\mathrm{g_c}(\Delta g) - \mu_{\mathrm{c}} \times ((n \times \Delta\,\mathrm{m}(\Delta g)) \operatorname{div} \mu_{\mathrm{m}}), \\ \\ (n \times \Delta\,\mathrm{m}(\Delta g)) \bmod \mu_{\mathrm{m}} \end{bmatrix}$$

The following three theorems state some interesting properties of the exponentiation function for genus intervals:

Theorem 502 *If*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

is a pitch system and Δg *is any genus interval in* ψ *then*

$$\iota_{\mathrm{g}}(\Delta g) = \epsilon_{\mathrm{g},-1}(\Delta g)$$

Theorem 503 *If*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

is a pitch system, $n_1, n_2, \ldots n_k$ is a collection of integers and Δg is a genus interval in ψ then

$$\epsilon_{\mathrm{g},n_k}\left(\ldots\epsilon_{\mathrm{g},n_2}\left(\epsilon_{\mathrm{g},n_1}\left(\Delta g\right)\right)\ldots\right)=\epsilon_{\mathrm{g},\prod_{j=1}^{k} n_j}\left(\Delta g\right)$$

Theorem 508 *If*

$$\psi=\left[\mu_{\mathrm{c}},\mu_{\mathrm{m}},f_0,p_{\mathrm{c},0}\right]$$

is a pitch system, $n_1, n_2, \ldots n_k$ is a collection of integers and Δg is a genus interval in ψ then

$$\sigma_{\mathrm{g}}\left(\epsilon_{\mathrm{g},n_1}\left(\Delta g\right),\epsilon_{\mathrm{g},n_2}\left(\Delta g\right),\ldots,\epsilon_{\mathrm{g},n_k}\left(\Delta g\right)\right)=\epsilon_{\mathrm{g},\sum_{j=1}^{k} n_j}\left(\Delta g\right)$$

1.3.8 Exponentiation of the genus transposition function

It is useful to define the concept of *exponentiating the genus transposition function.* This concept is defined as follows:

Definition 509 (Definition of $\tau_{\mathrm{g},n}\left(g,\Delta g\right)$) *If ψ is a pitch system and g is a genus in ψ and Δg is a genus interval in ψ then*

$$\tau_{\mathrm{g},n}\left(g,\Delta g\right)=\tau_{\mathrm{g}}\left(g,\epsilon_{\mathrm{g},n}\left(\Delta g\right)\right)$$

This definition, in combination with a number of other *MIPS* definitions and theorems can be used to prove the following theorem:

Theorem 510 *If*

$$\psi=\left[\mu_{\mathrm{c}},\mu_{\mathrm{m}},f_0,p_{\mathrm{c},0}\right]$$

is a pitch system, $n_1, n_2, \ldots, n_k$ is a collection of integers, g is a genus in ψ and Δg is a genus interval in ψ then

$$\tau_{\mathrm{g},n_k}\left(\ldots\tau_{\mathrm{g},n_2}\left(\tau_{\mathrm{g},n_1}\left(g,\Delta g\right),\Delta g\right)\ldots,\Delta g\right)=\tau_{\mathrm{g},\sum_{j=1}^{k} n_j}\left(g,\Delta g\right)$$

1.4 Using *MIPS* to model the A.S.A. pitch naming system and the Western tonal system of pitch interval names

The concepts introduced above can be used to construct four useful algorithms:

1. an algorithm that takes a *MIPS* pitch in ψ_{W} as input and generates the A.S.A. pitch name that corresponds to that pitch as output;

2. an algorithm that takes an A.S.A. pitch name as input and generates as output the *MIPS* pitch in ψ_{W} that corresponds to that pitch name;

3. an algorithm that takes a normal Western tonal pitch interval name as input (e.g. "Rising major third") and generates the corresponding pitch interval in ψ_{W} as output; and

4. an algorithm that takes a pitch interval in ψ_{W} as input and generates the normal Western tonal pitch interval name as output.

This section is devoted to describing these four algorithms.

1.4.1 Using the *MIPS* concept of a pitch to model the A.S.A. pitch naming system

As already mentioned above, in the A.S.A. pitch-naming system, a note has a *letter-name* (A to G), an *inflection* ($\ldots, \flat\flat, \flat, \natural, \sharp, \sharp\sharp, \ldots$) and an *octave number* (for example, middle C—$C\natural_4$—has an octave number of 4 and the C above middle C ($C\natural_5$) has an octave number of 5). This naming system derives from the staff notation system which has evolved over the past four hundred years or so to be a highly effective means of notating Western tonal music. To this extent, the pitch-naming system correctly models the Western tonal pitch system.

There is a one-to-one correspondence between a pitch in ψ_{W} (see Equation 1.1 above) and an A.S.A. pitch-name. Two algorithms can therefore be defined: one for returning the A.S.A. pitch-name that corresponds to any particular pitch; and another for returning the pitch that corresponds to any given A.S.A. pitch-name. The first of these algorithms uses the concept of chromatic genus defined above (see Definition 82).

Before describing these algorithms, it is necessary to define the concept of *concatenation* with respect to strings of characters. Let a string a be any sequence of characters $a_1 a_2 \ldots a_m$ and let b be any string $b_1 b_2 \ldots b_n$. The *concatenation* of b onto a, denoted $a \oplus b$, is equal to the string $a_1 a_2 \ldots a_m b_1 b_2 \ldots b_n$. The operation of concatenation on strings is associative: that is, for any three strings, a, b and c,

$$a \oplus (b \oplus c) = (a \oplus b) \oplus c$$

Both of these expressions can therefore be written $a \oplus b \oplus c$ without ambiguity.

The following algorithm, which will be called the `p-pn` algorithm, returns the A.S.A. pitch-name that corresponds to any given pitch:

1. Let p be a pitch in the pitch system ψ_{W}. For example, assume $p = [52, 34]$ (see Figure 1.10).
2. Let m be a numerical value used to represent the morph of p and set m to equal the value $\mathrm{m}(p)$. For example, if $p = [52, 34]$ then m would be made equal to 6.
3. Let l be a string of characters that is used to represent the letter-name of the A.S.A. pitch-name. Let l become equal to the value given in the second row of the following table that corresponds to the value of m.

m	0	1	2	3	4	5	6
l	"A"	"B"	"C"	"D"	"E"	"F"	"G"

   For example, if $m = 6$ then l will be made equal to "G".
4. Let g_{c} become equal to $\mathrm{g_c}(p)$. For example, if $p = [52, 34]$ then g_{c} would be made equal to 4.
5. Let c' become equal to the value in the second row of the following table that corresponds to the value of m.

m	0	1	2	3	4	5	6
c'	0	2	3	5	7	8	10

   The second row in this table gives, in order, the chroma of $A\natural, B\natural, \ldots G\natural$. In our example, $m = 6$ so c' will be made equal to 10.

6. Find the value $e = g_c - c'$. (For $p = [52, 34]$, $g_c = 4$ and $c' = 10$ therefore e would be made equal to -6.) If $e = 0$, this implies that the note is a natural note—that is, no sharps and no flats. If $e > 0$ then the note has e sharps and if $e < 0$ then the note has $-e$ flats.

7. Let i be a string of characters that is used to represent the inflection of the A.S.A. pitch-name. If $e = 0$ then let i become equal to the string "n". If $e > 0$ then let i become equal to a string consisting of e 's' characters (for example, if $e = 3$ then i should become equal to the string "sss"). If $e < 0$ then let i become equal to a string consisting of $-e$ 'f' characters (for example, if $e = -3$ then i should become equal to "fff".)[11]

8. Let o_m become equal to $o_m(p)$. If m is 0 or 1 then let $o_{A.S.A.}$ become equal to o_m. Otherwise, let $o_{A.S.A.}$ become equal to $o_m + 1$.

9. Let o become equal to the string of characters that represents in decimal the value of $o_{A.S.A.}$. For example, if $o_{A.S.A.} = 3$ then o should become equal to the string "3" and if $o_{A.S.A.} = -6$ then o should become equal to the string "-6".

10. Let n become equal to the string $l \oplus i \oplus o$ and output n. For example, for $p = [52, 34]$, l would be "G", i would be "ffffff" and o would be "5" giving a value for n of "Gffffff5" which is the desired result.

The Lisp function `p-pn` in Chapter 2 is an implementation of the `p-pn` algorithm. The following table gives some examples of the output generated by `p-pn` for a number of input pitches:

p	$[0,0]$	$[-1,0]$	$[0,-1]$	$[-9,-5]$	$[-10,-5]$	$[-9,-6]$	$[39,23]$	$[52,27]$	$[52,34]$	$[39,22]$	$[38,23]$
n	"An0"	"Af0"	"Gss0"	"Cn0"	"Cf0"	"Bs-1"	"Cn4"	"Gssssss4"	"Gffffff5"	"Bs3"	"Cf4"

The actual Lisp function call evaluated to generate these values looked like this in the Lisp Listener:

```
? (mapcar #'p-pn
        '((0 0) (-1 0) (0 -1) (-9 -5) (-10 -5) (-9 -6) (39 23) (52 27) (52 34) (39 22) (38 23)))
("An0" "Af0" "Gss0" "Cn0" "Cf0" "Bs-1" "Cn4" "Gssssss4" "Gffffff5" "Bs3" "Cf4")
?
```

The following algorithm performs the reverse process: when given an A.S.A. pitch-name n as input in the form of a string of the type generated as output by the `p-pn` algorithm just described, the following algorithm calculates the *MIPS* pitch that corresponds to the pitch-name n. The following algorithm is called the `pn-p` algorithm.

1. Let n be a string of characters representing a pitch-name (e.g. "Cn4", "Gssssss4", "Bf3").

2. If k is a string of characters then let $|k|$ be equal to the length of k (that is, the number of characters in k.)

3. Let l be the string that only contains the first character in the string n. So, for example, if n is "Gssssss4" then l will be equal to "G", if n is "Cn4" then l will be equal to "C".

[11] In the algorithm descriptions, characters will be enclosed between single quotes (e.g. 's', 'f') and strings will be enclosed by double quotes (e.g. "sss", "fff").

4. Let $n[x]$ return the xth character in the string n. For example, if n is equal to "Cn4" then $n[2]$ would be equal to the character 'n'.

5. Let i be the string that is constructed using the following procedure:

   (a) Let i become equal to the empty string, "".

   (b) Let x become equal to 2.

   (c) Let j become equal to the string that consists of the single character $n[x]$.

   (d) Let i become equal to $i \oplus j$.

   (e) Let x become equal to $x + 1$.

   (f) If $n[x]$ is a member of the set of characters

   {'-','1','2','3','4','5','6','7','8','9'}

   or if x is greater than the length of n then go to step 6 and return i. Otherwise go to step 5c.

6. If i is equal to the string "n" or a string consisting entirely of 's' characters (e.g. "sssss") or a string consisting entirely of 'f' characters ("fffff") then go to step 7. Otherwise return an error.

7. Let o become equal to the string that is returned by the following procedure:

   (a) Let y become equal to the length of i.

   (b) Let x become equal to $y + 2$.

   (c) Let o become equal to the string that contains the single character $n[x]$.

   (d) Let x become equal to $x + 1$.

   (e) If $n[x]$ exists then let j become equal to the string that consists of the single character $n[x]$. Otherwise let j become equal to the empty string "".

   (f) If j is non-empty then let o become equal to $o \oplus j$.

   (g) If j is non-empty then go to step 7d. Otherwise go to step 8 and return o.

8. Let $o_{A.S.A.}$ become equal to the decimal value expressed by the string o. For example, if o is equal to the string "-23" then $o_{A.S.A}$ would become equal to -23.

9. Let m become equal to the value in the second row of the following table that corresponds to the value of l.

l	"A"	"B"	"C"	"D"	"E"	"F"	"G"
m	0	1	2	3	4	5	6

10. Let c' be made equal to the value in the second row of the following table that corresponds to the value of m.

m	0	1	2	3	4	5	6
c'	0	2	3	5	7	8	10

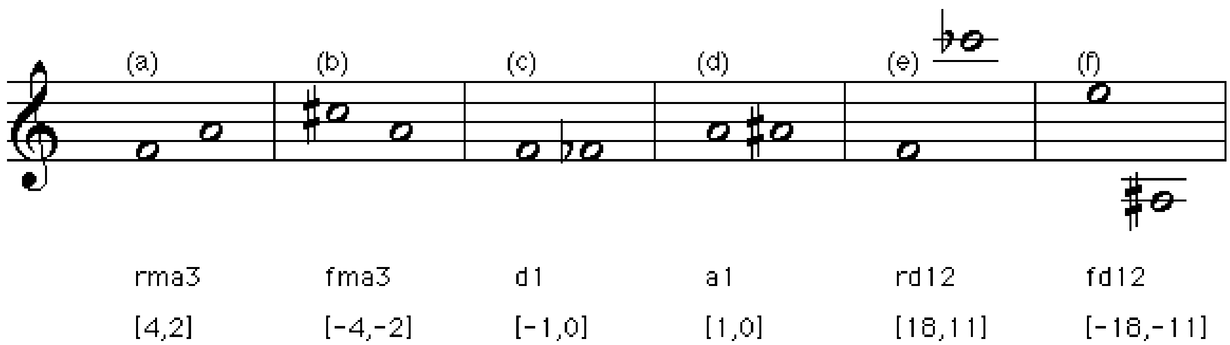


Figure 1.11: Pitch intervals and pitch interval names.

11. If i is equal to "n" then let e become equal to 0. If i is a string of 'f' characters (e.g. "fff") then let e become equal to the value $-1 \times |i|$. If i is a string of 's' characters then let e become equal to the value $|i|$.

12. If m is 0 or 1, then let o_{m} become equal to $o_{A.S.A.}$. Otherwise let o_{m} become equal to $o_{A.S.A.} - 1$.

13. Let p_{c}, the chromatic pitch of the pitch that will be generated as output, become equal to the value $e + c' + \mu_{\mathrm{c}} \times o_{\mathrm{m}}$ where μ_{c} is the chromatic modulus of the pitch system ψ_{W}, that is, $\mu_{\mathrm{c}} = 12$.

14. Let p_{m}, the morphetic pitch of the pitch that will be generated as output, become equal to the value $o_{\mathrm{m}} \times \mu_{\mathrm{m}} + m$ where μ_{m} is the morphetic modulus of the pitch system ψ_{W}, that is, $\mu_{\mathrm{m}} = 7$.

15. Let p become equal to the ordered pair, $[p_{\mathrm{c}}, p_{\mathrm{m}}]$ and output p.

The Lisp function `pn-p` in Chapter 2 is an implementation of the `pn-p` algorithm. The following table gives some examples of the output generated by `p-pn` for a number of input pitch names:

n	"An0"	"Af0"	"Gss0"	"Cn0"	"Cf0"	"Bs-1"	"Cn4"	"Gssssss4"	"Gffffff5"	"Bs3"	"Cf4"
p	$[0,0]$	$[-1,0]$	$[0,-1]$	$[-9,-5]$	$[-10,-5]$	$[-9,-6]$	$[39,23]$	$[52,27]$	$[52,34]$	$[39,22]$	$[38,23]$

The actual Lisp function call evaluated to generate these values looked like this in the Lisp Listener:

```
? (mapcar #'pn-p
        '("An0" "Af0" "Gss0" "Cn0" "Cf0" "Bs-1" "Cn4" "Gssssss4" "Gffffff5" "Bs3" "Cf4"))
((0 0) (-1 0) (0 -1) (-9 -5) (-10 -5) (-9 -6) (39 23) (52 27) (52 34) (39 22) (38 23))
?
```

1.4.2 Using the *MIPS* concept of a pitch interval to model the Western tonal pitch interval naming system

Figure 1.11 shows a number of pairs of notes and written beneath each pair is a code which is an abbreviation for the traditional pitch interval name for the pitch interval from the first note in the pair to the second note.

Direction	*Abbreviation*
rising	r
falling	f

Type	*Abbreviation*
perfect	p
major	ma
minor	mi
augmented	a
double-augmented	aa
triple-augmented	aaa
...	...
diminished	d
double-diminished	dd
triple-diminished	ddd
...	...

Size	*Abbreviation*
prime	1
second	2
third	3
fourth	4
...	...

Table 1.1: Code for abbreviated notation of traditional Western tonal pitch interval names.

A pitch interval name in the traditional Western tonal pitch interval naming system has three parts: a *direction* which can either be rising or falling[12]; a *type* which is a member of the infinite set,

{..., double-augmented, augmented, major, perfect, minor, diminished, double-diminished,...}

and a *size* which is a member of the set

{prime, second, third, fourth, fifth, sixth, seventh, octave, ninth, tenth,...}

In this document, an abbreviated format will be used to denote traditional pitch interval names. Table 1.1 describes this abbreviated notation. For example, a rising major third would be denoted 'rma3', a falling double-diminished sixth would be denoted 'fdd6' and a perfect prime would be denoted 'p1'.

There is a one-to-one correspondence between a pitch interval name in the traditional Western tonal pitch-naming system and a *MIPS* pitch interval in the pitch system ψ_{W} (see Equation 1.1). In Figure 1.11 each pair of notes has written beneath it the traditional pitch name in abbreviated format together with the pitch interval in ψ_{W} that corresponds to that pitch name. As can be seen in Figure 1.11, the chromatic pitch interval associated with the interval gives the change in chromatic pitch and the morphetic pitch interval

[12]The interval of a prime does not have a direction because it does not result in a change in morphetic pitch.

gives the change in morphetic pitch (i.e. the number of steps moved on the staff). A positive chromatic or morphetic pitch interval corresponds to an increase in chromatic or morphetic pitch respectively. In Figure 1.11, intervals (b), (d) and (f) are the inverses of intervals (a), (c) and (e) respectively.

The remainder of this section will be devoted to describing two algorithms. The first one, called `pi-pin`, takes as input a pitch interval Δp in ψ_{W} and generates as output the traditional pitch interval name that corresponds to Δp. The second algorithm, `pin-pi`, performs the reverse process: when given as input a pitch name Δn it generates as output the corresponding pitch interval in ψ_{W}.

Before presenting these algorithms, it is necessary to define a function that returns the *chromatic genus interval of a pitch interval*, denoted $\Delta\,\mathrm{g_c}\,(\Delta p)$. This concept is defined as follows:

Definition 279 (Chromatic genus interval of a pitch interval) *If p_1 and p_2 are any two pitches in a pitch system ψ then*

$$\Delta p = \Delta\,\mathrm{p}\,(p_1, p_2) \Rightarrow \Delta\,\mathrm{g_c}\,(\Delta p) = \Delta\,\mathrm{g_c}\,(p_1, p_2)$$

This definition along with other definitions and theorems in *MIPS* can be used to prove the following theorem which provides us with a formula for calculating the chromatic genus interval of a pitch interval:

Theorem 280 (Formula for $\Delta\,\mathrm{g_c}\,(\Delta p)$) *If Δp is a pitch interval in*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

then:

$$\Delta\,\mathrm{g_c}\,(\Delta p) = \Delta\,\mathrm{p_c}\,(\Delta p) - \mu_{\mathrm{c}} \times (\Delta\,\mathrm{p_m}\,(\Delta p)\ \mathrm{div}\ \mu_{\mathrm{m}})$$

The algorithm `pi-pin` takes the following form:

1. Let Δp be a pitch interval in ψ_{W}.

2. Let d be a string that will be used to represent the direction of the pitch interval name. If $\Delta\,\mathrm{p_m}\,(\Delta p) = 0$ then let d be made equal to the empty string “”. If $\Delta\,\mathrm{p_m}\,(\Delta p) > 0$ then d should be made equal to the string “r”. If $\Delta\,\mathrm{p_m}\,(\Delta p) < 0$ then d should be made equal to the string “f”.

3. Let s' be made equal to the value $\mathrm{abs}\,(\Delta\,\mathrm{p_m}\,(\Delta p)) + 1$ and let s, the string that will represent the size of the pitch interval name generated as output, be made equal to the string that represents in decimal format the value of s'. For example, if $s' = 3$ then s will be made equal to the string “3”.

4. Let $\Delta m'$ be made equal to the value $\mathrm{abs}\,(\Delta\,\mathrm{p_m}\,(\Delta p))\ \mathrm{mod}\ \mu_{\mathrm{m}}$ where μ_{m} is the morphetic modulus which in the case of ψ_{W} is equal to 7.

5. Let $\Delta c'$ become equal to the value in the second row of the following table that corresponds to the value of $\Delta m'$ in the top row.

$\Delta m'$	0	1	2	3	4	5	6
$\Delta c'$	0	2	4	5	7	9	11

6. Let t' become equal to the value in the second row of the following table that corresponds to the value of $\Delta m'$ in the top row.

$\Delta m'$	0	1	2	3	4	5	6
t'	"p"	"ma"	"ma"	"p"	"p"	"ma"	"ma"

7. If $\Delta\,\mathrm{p_m}\,(\Delta p) \geq 0$ then let e be made equal to the value $\Delta\,\mathrm{g_c}\,(\Delta p) - \Delta c'$. Otherwise, let e become equal to $\Delta\,\mathrm{g_c}\,(\iota_\mathrm{p}\,(\Delta p)) - \Delta c'$.

8. (a) If t' is equal to the string "p" and $e = 0$ then let t become equal to the string "p".
   (b) If t' is equal to the string "p" and $e > 0$ then let t become equal to the string that consists of e 'a' characters. (For example, if $e = 3$ then t should be made equal to "aaa".)
   (c) If t' is equal to "p" and $e < 0$ then let t become equal to the string that consists of $-e$ 'd' characters. (For example, if $e = -3$ then t should be made equal to "ddd".)
   (d) If t' is equal to "ma" and $e = 0$ then let t become equal to "ma".
   (e) If t' is equal to "ma" and $e = -1$ then let t become equal to "mi".
   (f) If t' is equal to "ma" and $e < -1$ then let t become equal to the string that consists of $-e - 1$ 'd' characters. (For example, if $e = -4$ then t should be made equal to "ddd".)
   (g) If t' is equal to "ma" and $e > 0$ then let t become equal to the string that consists of e 'a' characters. (For example, if $e = 2$ then t should be made equal to "aa".)

9. Let Δn become equal to the string $d \oplus t \oplus s$ and generate Δn as output.

The Lisp function `pi-pin` in Chapter 2 is an implementation of the `pi-pin` algorithm. The following table gives some examples of the output generated by `pi-pin` for a number of input pitch intervals:

Δp	$[2,1]$	$[3,1]$	$[0,1]$,	$[-1,1]$	$[-7,-4]$	$[-6,-4]$	$[-17,-10]$	$[0,7]$	$[-1,0]$	$[1,0]$
Δn	"rma2"	"ra2"	"rd2"	"rdd2"	"fp5"	"fd5"	"fp11"	"rddddddddddddd8"	"d1"	"a1"

The actual Lisp function call evaluated to generate these values looked like this in the Lisp Listener:

```
? (mapcar #'pi-pin
        '((2 1) (3 1) (0 1) (-1 1) (-7 -4) (-6 -4) (-17 -10) (0 7) (-1 0) (1 0)))
("rma2" "ra2" "rd2" "rdd2" "fp5" "fd5" "fp11" "rddddddddddddd8" "d1" "a1")
?
```

The algorithm `pin-pi` performs the reverse task to `pi-pin`: it takes a traditional Western tonal pitch interval name as input and generates as output the pitch interval in ψ_W that corresponds to that pitch interval name. This algorithm takes the following form:

1. Let Δn be a string that represents a pitch interval name such as "rma3", "fd11", "d1" etc.

2. If the first character in Δn is a member of the set {'r','f'} then let d be the string that contains only the first character in Δn. Otherwise, let d be made equal to the empty string, "". For example, if Δn is "rma3" then d should be made equal to the string "r"; if Δn is "fmi6" then d should be made equal to the string "f"; and if Δn is "p1" then d should be made equal to the string "".

3. If d is equal to the empty string, then let t be made equal to the substring of Δn that begins with the first character in Δn and ends with the character that precedes the earliest character in the string that is a member of the set
$$\{\text{'1'},\text{'2'},\text{'3'},\text{'4'},\text{'5'},\text{'6'},\text{'7'},\text{'8'},\text{'9'}\}$$
For example, if Δn is equal to "ddd1" then t should be made equal to the string "ddd". If d is a member of the set {"r","f"} then let t be made equal to the substring of Δn that begins with the second character in Δn and ends with the character that precedes the earliest character in Δn that is a member of the set
$$\{\text{'1'},\text{'2'},\text{'3'},\text{'4'},\text{'5'},\text{'6'},\text{'7'},\text{'8'},\text{'9'}\}$$
For example, if Δn is equal to "rma3" then t should be made equal to the string "ma".

4. If t is not a member of the set
$$\{\text{"p"},\text{"ma"},\text{"mi"}\}$$
and t is not a string that only contains 'd' characters (e.g. "ddd") and t is not a string that contains only 'a' characters (e.g. "aaa") then stop the algorithm and return an error. Otherwise, go on to the next step.

5. Let s be the substring of Δn that begins with the first character in Δn that is a member of the set
$$\{\text{'1'},\text{'2'},\text{'3'},\text{'4'},\text{'5'},\text{'6'},\text{'7'},\text{'8'},\text{'9'}\}$$
and ends with the last character in Δn. For example, if Δn is equal to "rma10" then s should be made equal to the string "10".

6. If s is a non-empty string that only contains characters that are members of the set
$$\{\text{'1'},\text{'2'},\text{'3'},\text{'4'},\text{'5'},\text{'6'},\text{'7'},\text{'8'},\text{'9'}\}$$
then go on to the next step. Otherwise stop and return an error.

7. Let s' be made equal to the decimal value represented by the string s. For example, if s is the string "12" then s' would be made equal to the value 12.

8. If d is equal to the string "f" then Δp_{m} should be made equal to the value $1 - s'$ otherwise, Δp_{m} should be made equal to the value $s' - 1$.

9. Let $\Delta m'$ be made equal to the value $\mathrm{abs}\,(\Delta p_{\mathrm{m}}) \bmod \mu_{\mathrm{m}}$ where μ_{m} is the morphetic modulus which in the case of ψ_{W} is equal to 7.

10. Let $\Delta c'$ be made equal to the value in the second row of the following table that corresponds to the value of $\Delta m'$ found in the previous step.

$\Delta m'$	0	1	2	3	4	5	6
$\Delta c'$	0	2	4	5	7	9	11

11. Let $\Delta p_{\mathrm{c},1}$ be made equal to the value
$$\Delta c' + \mu_{\mathrm{c}} \times (\mathrm{abs}\,(\Delta p_{\mathrm{m}})\ \mathrm{div}\ \mu_{\mathrm{m}})$$

12. Let t' be made equal to the value in the table that corresponds to the value of $\Delta m'$ found in step 9:

$\Delta m'$	0	1	2	3	4	5	6
t'	"p"	"ma"	"ma"	"p"	"p"	"ma"	"ma"

13. (a) If t' is equal to the string "p" and t is also equal to the string "p" then let e become equal to 0.
    (b) If t' is equal to the string "p" and t is a string that consists entirely of 'd' characters (e.g. "ddd") then let e become equal to $-1 \times |t|$.
    (c) If t' is equal to "p" and "t" is equal to a string that consists entirely of 'a' characters (e.g. "aaa") then let e become equal to $|t|$.
    (d) If t' is equal to "ma" and t is equal to "ma" then let e become equal to 0.
    (e) If t' is equal to "ma" and t is equal to "mi" then let e become equal to -1.
    (f) If t' is equal to "ma" and t is equal to a string that consists entirely of 'd' characters then let e become equal to $-1 \times (|t| + 1)$.
    (g) If t' is equal to "ma" and t is equal to a string that consists entirely of 'a' characters then let e become equal to $|t|$.

14. If $\Delta p_{\mathrm{m}} < 0$ then let Δp_{c} become equal to the value

$$-1 \times (\Delta p_{\mathrm{c},1} + e)$$

otherwise let Δp_{c} become equal to the value $\Delta p_{\mathrm{c},1} + e$.

15. Let Δp become equal to the ordered pair $[\Delta p_{\mathrm{c}}, \Delta p_{\mathrm{m}}]$ and return the value Δp.

The Lisp function `pin-pi` in Chapter 2 is an implementation of the `pin-pi` algorithm. The following table gives some examples of the output generated by `pin-pi` for a number of input pitch interval names:

Δn	"rma2"	"ra2"	"rd2"	"rdd2"	"fp5"	"fd5"	"fp11"	"rddddddddddddd8"	"d1"	"a1"
Δp	$[2, 1]$	$[3, 1]$	$[0, 1]$,	$[-1, 1]$	$[-7, -4]$	$[-6, -4]$	$[-17, -10]$	$[0, 7]$	$[-1, 0]$	$[1, 0]$

The actual Lisp function call evaluated to generate these values looked like this in the Lisp Listener:

```
? (mapcar #'pin-pi
        '(rma2 ra2 rd2 rdd2 fp5 fd5 fp11 rddddddddddddd8 d1 a1))
((2 1) (3 1) (0 1) (-1 1) (-7 -4) (-6 -4) (-17 -10) (0 7) (-1 0) (1 0))
?
```

1.5 Summary

1. *MIPS* is a formal language invented by the author that is designed to be used for investigating the mathematical properties of pitch systems and collections of pitches within those systems.

2. *MIPS* is based on two fundamental concepts: the concept of a *pitch system* and the concept of a *pitch*.

3. A *MIPS* pitch system,
$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$
models a pitch system that employs scales containing μ_{m} notes, performed in an equal-tempered tuning system where the frequency f_0 is associated with the chromatic pitch $p_{\mathrm{c},0}$ and where the octave is divided into μ_{c} equal frequency intervals.

4. In principle, if the frequency of a pitch within a pitch system can be calculated from its *MIPS* pitch, then the pitch system can be modelled in *MIPS* (provided that one defines an appropriate frequency function in place of that given in Definition 66). This provides a way for modelling non-equal-tempered pitch systems in *MIPS*.

5. *MIPS* is constructed around four mathematical representations of octave equivalence: chroma, morph, chromamorph and genus. The chroma, morph and chromamorph representations have been used elsewhere but the genus representation is presented here for the first time. The concepts of chroma, morph and chromamorph fail to model correctly the traditional tonal concept of octave equivalence. However, the genus representation of octave equivalence not only correctly models the traditional tonal concept but also can be generalised to any other pitch system without first having to know which sets in that pitch system correspond to the diatonic sets of the Western pitch system.

6. Definitions and formulae have been given for deriving the chroma, morph, chromatic genus and chromamorph of a genus. Formulae and theorems have also been provided for transposing a genus by a genus interval and for summing, inverting and exponentiating genus intervals. Many more concepts and formulae relating to the genus representation of octave equivalence (including formulae for manipulating genus sets and genus interval sets) can be found in Chapter 4.

7. Two algorithms, `pn-p` and `p-pn`, were presented for converting between A.S.A. pitch names and *MIPS* pitches in the pitch system ψ_{W}.

8. Two algorithms, `pin-pi` and `pi-pin`, were presented for converting between Western tonal pitch interval names (e.g. "rma3") and *MIPS* pitch intervals.

9. All the theorems in this chapter have been presented without proof. However, all the theorems in this chapter are proved in Chapter 4.

Chapter 2

Lisp implementation of the algorithms `p-pn`, `pn-p`, `pi-pin` and `pin-pi`

Given below is the full Lisp source code for implementations of the algorithms `p-pn`, `pn-p`, `pi-pin` and `pin-pi` described in sections 1.4.1 and 1.4.2 above.

```
#|
Algorithms for converting between A.S.A. pitch names and MIPS pitches.
|#
(setf *save-local-symbols* t)
(setf *verbose-eval-selection* t)

(defvar mum 7)
(setf mum 7)
(defvar muc 12)
(setf muc 12)

(defun p-pn (p)
  (let* ((m (p-m p))
         (l (elt '("A" "B" "C" "D" "E" "F" "G") m))
         (gc (p-gc p))
         (cdash (elt '(0 2 3 5 7 8 10) m))
         (e (- gc cdash))
         (i "")
         (i (cond ((< e 0) (dotimes (j (- e) i) (setf i (concatenate 'string i "f"))))
                  ((> e 0) (dotimes (j e i) (setf i (concatenate 'string i "s"))))
                  ((= e 0) "n")))
         (om (p-om p))
         (oasa (if (or (= m 0) (= m 1))
                   om
                   (+ 1 om)))
         (o (format nil "~D" oasa)))
    (concatenate 'string l i o)))
```

```
(defun p-m (p)
  (bmod (p-pm p) mum))

(defun bmod (x y)
  (- x
     (* y
        (int (/ x y)))))

(defun p-pm (p)
  (second p))

(defun int (x)
  (values (floor x)))

(defun p-gc (p)
  (- (p-pc p)
     (* muc (p-om p))))

(defun p-pc (p)
  (first p))

(defun p-om (p)
  (div (p-pm p) mum))

(defun div (x y)
  (int (/ x y)))

(defun pn-p (pn-as-input)
  (let* ((n (if (stringp pn-as-input)
              (string-upcase pn-as-input)
              (string-upcase (string pn-as-input))))
         (l (string (elt n 0)))
         (i (do* ((i "")
                  (x 2)
                  (j (string (elt n (- x 1))) (string (elt n (- x 1))))
                  (i (concatenate 'string i j) (concatenate 'string i j))
                  (x (+ 1 x) (+ 1 x)))
                 ((or (>= x (length n))
                      (member (elt n (- x 1)) '(#\- #\1 #\2 #\3 #\4 #\5 #\6 #\7 #\8 #\9)))
                  i)))
         (is-good-i (well-formed-inflection-p i))
         (o (if is-good-i
              (do* ((y (length i))
                    (x (+ y 2))
```

```
                    (o (string (elt n (- x 1))))
                    (x (+ 1 x) (+ 1 x))
                    (j (if (<= x (length n))
                         (string (elt n (- x 1)))
                         "")
                       (if (<= x (length n))
                         (string (elt n (- x 1)))
                         ""))
                    (o (if (equalp j "") o
                           (concatenate 'string o j))
                       (if (equalp j "") o
                           (concatenate 'string o j))))
                   ((equalp j "")
                    o))))
         (oasa (if is-good-i (string-to-number o)))
         (m (if is-good-i (position l
                                  '("A" "B" "C" "D" "E" "F" "G")
                                  :test #'equalp)))
         (cdash (if is-good-i (elt '(0 2 3 5 7 8 10) m)))
         (e (if is-good-i (cond ((equalp i "N") 0)
                              ((equalp (elt i 0) #\F) (* -1 (length i)))
                              ((equalp (elt i 0) #\S) (length i)))))
         (om (if is-good-i (if (or (= m 1) (= m 0))
                             oasa (- oasa 1))))
         (pc (if is-good-i (+ e cdash (* muc om))))
         (pm (if is-good-i (+ m (* om mum)))))
    (if is-good-i (list pc pm))))

(defun string-to-number (s)
  (if (well-formed-number-string-p s)
    (if (string-is-negative-p s)
      (let ((n 0))
        (dotimes (i (- (length s) 1) (* -1 n))
          (setf n (+ (* 10 n)
                     (- (char-code (elt s (+ 1 i)))
                        (char-code #\0))))))
      (let ((n 0))
        (dotimes (i (length s) n)
          (setf n (+ (* 10 n)
                     (- (char-code (elt s i))
                        (char-code #\0)))))))))

(defun string-is-negative-p (s)
  (equalp #\- (char s 0)))
```

```
;(string-is-negative-p "23")

(defun well-formed-number-string-p (s)
  (let ((wf t))
    (dotimes (i (length s) wf)
      (if (not (or (<= (char-code #\0) (char-code (char s i)) (char-code #\9))
                   (and (= i 0)
                        (equalp (char s i) #\-))))
        (setf wf nil)))))

#|
(well-formed-number-string-p "23")
|#

(defun well-formed-inflection-p (i)
  (or (equalp i "N")
      (let ((wf t))
        (dotimes (j (length i) wf)
          (if (not (equalp (char i j) #\F))
            (setf wf nil))))
      (let ((wf t))
        (dotimes (j (length i) wf)
          (if (not (equalp (char i j) #\S))
            (setf wf nil))))))

#|
TESTS FOR p-pn and pn-p

(mapcar #'p-pn
        '((0 0) (-1 0) (0 -1) (-9 -5) (-10 -5) (-9 -6) (39 23) (52 27) (52 34) (39 22) (38 23)))

(mapcar #'pn-p
        '("An0" "Af0" "Gss0" "Cn0" "Cf0" "Bs-1" "Cn4" "Gssssss4" "Gffffff5" "Bs3" "Cf4"))
|#

(defun pi-pin (pint)
  (let* ((pmint (p-int-pm-int pint))
         (d (cond ((= 0 pmint) "")
                  ((> pmint 0) "r")
                  ((< pmint 0) "f")))
         (sdash (+ 1 (abs pmint)))
         (s (format nil "~D" sdash))
         (mintdash (bmod (abs pmint) mum))
         (cintdash (elt '(0 2 4 5 7 9 11) mintdash))
         (tdash (elt '("p" "ma" "ma" "p" "p" "ma" "ma") mintdash))
```

```
         (e (if (>= pmint 0) (- (p-int-gc-int pint) cintdash) (- (p-int-gc-int (invp pint)) cintdash)))
         (ty (cond ((and (equalp tdash "p") (= e 0))
                    "p")
                   ((and (equalp tdash "p") (> e 0))
                    (let ((x "")) (dotimes (i e x) (setf x (concatenate 'string x "a")))))
                   ((and (equalp tdash "p") (< e 0))
                    (let ((x "")) (dotimes (i (- e) x) (setf x (concatenate 'string x "d")))))
                   ((and (equalp tdash "ma") (= e 0))
                    "ma")
                   ((and (equalp tdash "ma") (= e -1))
                    "mi")
                   ((and (equalp tdash "ma") (< e -1))
                    (let ((x "")) (dotimes (i (- (- e) 1) x) (setf x (concatenate 'string x "d")))))
                   ((and (equalp tdash "ma") (> e 0))
                    (let ((x "")) (dotimes (i e x) (setf x (concatenate 'string x "a"))))))))
    (concatenate 'string d ty s)))

(defun p-int-pm-int (pint)
  (second pint))

(defun p-int-gc-int (pint)
  (- (p-int-pc-int pint)
     (* muc
        (div (p-int-pm-int pint)
             mum))))

(defun p-int-pc-int (pint)
  (first pint))

(defun invp (pint)
  (list (- (p-int-pc-int pint))
        (- (p-int-pm-int pint))))

#|
Tests for pi-pin and pin-pi

(mapcar #'pi-pin
        '((0 0) (2 1) (1 1) (3 1) (0 1) (-1 1) (4 1) (-7 -4)
          (-6 -4) (-5 -4) (-17 -10) (0 7) (-1 0) (1 0)))
|#

(defun pin-pi (pitch-interval-name)
  (let* ((pin (if (stringp pitch-interval-name)
                (string-upcase pitch-interval-name)
                (string-upcase (string pitch-interval-name))))
```

```
(d (char pin 0))
(d (if (member d '(#\F #\R) :test #'equalp) (string d) ""))
(ty (do* ((ty "")
          (x (if (equalp d "") 0 1))
          (j (string (elt pin x)) (string (elt pin x)))
          (ty (concatenate 'string ty j) (concatenate 'string ty j))
          (x (+ 1 x) (+ 1 x)))
         ((or (>= x (length pin))
              (member (elt pin x) '(#\1 #\2 #\3 #\4 #\5 #\6 #\7 #\8 #\9)))
          ty)))
(ty-error (not (well-formed-interval-type-p ty)))
(s (if (not ty-error)
     (do* ((y (length ty))
           (x (if (equalp d "") y (+ y 1)))
           (s (string (elt pin x)))
           (x (+ 1 x) (+ 1 x))
           (j (if (< x (length pin))
                  (string (elt pin x))
                  "")
              (if (< x (length pin))
                  (string (elt pin x))
                  ""))
           (s (if (equalp j "") s
                  (concatenate 'string s j))
              (if (equalp j "") s
                  (concatenate 'string s j))))
          ((equalp j "")
           s))))
(s-error (if (not ty-error) (not (well-formed-number-string-p s))))
(s-dash (if (or s-error ty-error) nil (string-to-number s)))
(pmintvar (if (or s-error ty-error) nil (if (equalp d "f") (- 1 s-dash) (- s-dash 1))))
(mint-dash (if (or s-error ty-error) nil (bmod (abs pmintvar) mum)))
(cint-dash (if (or s-error ty-error) nil (elt '(0 2 4 5 7 9 11) mint-dash)))
(pcintone (if (or s-error ty-error) nil (+ cint-dash
                                            (* muc
                                               (div (abs pmintvar)
                                                    mum)))))
(t-dash (if (or s-error ty-error) nil (elt '("p" "ma" "ma" "p" "p" "ma" "ma") mint-dash)))
(e (if (or s-error ty-error) nil
       (cond ((and (equalp ty "p") (equalp t-dash "p")) 0)
             ((and (equalp t-dash "p") (equalp (char ty 0) #\D)) (* (- 1) (length ty)))
             ((and (equalp t-dash "p") (equalp (char ty 0) #\A)) (length ty))
             ((and (equalp ty "ma") (equalp t-dash "ma")) 0)
             ((and (equalp t-dash "ma") (equalp ty "mi")) (- 1))
             ((and (equalp t-dash "ma") (equalp (char ty 0) #\D)) (* (- 1)
```

```
                                                                                 (+ (length ty) 1)))
                      ((and (equalp t-dash "ma") (equalp (char ty 0) #\A)) (length ty)))))
         (pcintvar (if (or s-error ty-error) nil
                       (if (< pmintvar 0) (* (- 1) (+ e pcintone)) (+ e pcintone)))))
    (list pcintvar pmintvar)))

(defun well-formed-interval-type-p (ty)
  (or (member ty '("MA" "MI" "P") :test #'equalp)
      (let ((wf t))
        (dotimes (j (length ty) wf)
          (if (not (equalp (char ty j) #\D))
            (setf wf nil))))
      (let ((wf t))
        (dotimes (j (length ty) wf)
          (if (not (equalp (char ty j) #\A))
            (setf wf nil))))))

#|
(mapcar #'pin-pi
        '(rma2 ra2 rd2 rdd2 fp5 fd5 fp11 rddddddddddd8 d1 a1))
(pin-pi 'd1)
(setf pitch-interval-name 'd1)
|#
```

Chapter 3

How to read the tabular proofs

In this document the proof of each theorem is presented in the form of a table with four columns. For example, Table 3.1 shows the proof of Theorem 582.

Each row in the proof has a label of the form Rn which is given in the first column. Each row is either an inference, an assumption or a statement of a well-known mathematical result that is not proved within this document. In Table 3.1, rows R2, R3 and R4 are inferences and row R1 is an assumption.

If a row simply states a well-known mathematical result without proof then it will take the following form:

R3	$\sin^2 x + \cos^2 x = 1$

Such a row will consist of just two elements: the label of the row (in this case 'R3') in the first column of the table and the expression that states the mathematical result in the fourth column.

A row of the form of row R1 in Table 3.1 expresses a condition that is assumed to be true for the remainder of the proof in which the row occurs. A row that expresses an assumption consists of three elements: the first element is the label (e.g. 'R1') which occurs in the first column of the table; the second element consists of the word 'Let' which occurs in the second column of the table; and the third element is a statement of the condition that is assumed to be true (e.g. '$p = [p_\mathrm{c}, p_\mathrm{m}]$ is any pitch whatsoever in a pitch system ψ'). This statement occurs in the fourth column of the table.

A row of the form of R2 in Table 3.1 expresses an inference and consists of four elements. The first element is the label (e.g. 'R2') which occurs in the first column of the table. The second element is the list of premises which occurs in the second column of the table. The third element consists of the symbol '$\Rightarrow$' (implies) and occurs in the third column of the table. Finally, the fourth element consists of the conclusion

R1	Let		$p = [p_\mathrm{c}, p_\mathrm{m}]$ be any pitch whatsoever in a pitch system ψ.
R2	R1 & 62	$\Rightarrow$	p_c can only take any integer value.
R3	R1 & 62	$\Rightarrow$	p_m can only take any integer value.
R4	R2, R3 & 581	$\Rightarrow$	$\underline{p}_\mathrm{u} = \{[p_\mathrm{c}, p_\mathrm{m}] : p_\mathrm{c}, p_\mathrm{m} \in \mathbb{Z}\}$ where $\mathbb{Z}$ is the universal set of integers.

Table 3.1: Proof of Theorem 582

of the inference. Taken as a whole, an inference is a statement that the conclusion (the fourth element in the row) can be logically deduced from the list of premises (the second element in the row). The list of premises can contain two different types of element: the label of an earlier row in the current proof (e.g. R1 in the list of premises in row R2 in Table 3.1) or the reference number of a previous definition or theorem (e.g. the number 62 in the list of premises in row R2). Thus, the row R2 in Table 3.1 should be read: "The row R1 in this proof and Definition 62, taken together, logically imply that the value p_{c} may take any integer value."

In some cases, the conclusion of an inference is itself an implication. Consider, for example, the following row:

$$\text{R12} \quad \text{R3 \& 4} \quad \Rightarrow \quad x \Rightarrow y$$

This proof row states that line R3 in the current proof, taken with the previously stated theorem or definition whose reference number is 4 together imply that x implies y. Note that this row should *not* be understood to mean that line R3 and theorem/definition 4 together imply x *which in turn implies* y.

The definitions and theorems in the specification of *MIPS* given in Chapter 4 are numbered in the order in which they appear in the specification in one, single sequence—that is, the definitions are not numbered separately from the theorems. This means that any theorem or definition can be uniquely identified by its reference number—each theorem and definition has a unique number that it does not share with any other theorem or definition. For example, Theorem 582 has the number 582 which is unique to that theorem—no definition has the number 582 and no other theorem has this number.

The proofs are intended to be as easy to understand and as complete as possible. It should be possible for anyone with elementary school algebra (and enough patience) to be able to understand all the proofs.

Chapter 4

Formal specification of *MIPS*

4.1 Sets and ordered sets

4.1.1 Definitions of set and ordered set

Definition 1 (Universal set) *An object is a well-formed* universal set *if and only if it is a well-defined collection of objects that are all distinct in some specified way.*

Definition 2 (Universal set membership) *If S is a universal set then a is an* element *or* member *of S, denoted $a \in S$, if and only if a is equal to one of the objects in S. If a is not equal to any of the objects in S then one can say that a is not an element of S and denote this fact as follows: $a \notin S$.*

Definition 3 (Set) *An object is a well-formed* set *if and only if it is a collection of objects that are all distinct members of a single specified universal set. When written out in full, a set is enclosed within braces and the objects in the set are separated from each other by commas:*

$$S = \{s_1, s_2, \ldots\}$$

Definition 4 (Ordered set) *An object is a well-formed* ordered set *if and only if it is a collection of objects (not necessarily distinct and not necessarily all from the same universal set). When written out in full, an ordered set is enclosed in square brackets and the objects in the ordered set are separated from each other by commas:*

$$S = [s_1, s_2, \ldots]$$

Definition 5 (Set membership) *If S is a set or ordered set then a is an* element *or* member *of S, denoted $a \in S$, if and only if a is equal to one of the objects in S. If a is not equal to any member of S then one can say that a is not an element of S and denote this fact as follows: $a \notin S$.*

Definition 6 (Set order) *If S is a set or ordered set then the* order *or* cardinality *of S, denoted $|S|$, is equal to the number of elements in S.*

Definition 7 (Empty set) *The* empty set *is that unique set that contains no members. It is denoted $\emptyset$ or $\{\,\}$.*

Definition 8 (Empty ordered set) *The* empty ordered set *is that unique ordered set that contains no members. It is denoted $[\,]$.*

4.1.2 Operations on ordered sets

Definition 9 (Element of an ordered set) *If S is an ordered set,*

$$S = [s_1, s_2, \dots s_k, \dots]$$

then, by definition,

$$\mathrm{e}\,(S, k) = s_k$$

for all integer k such that $1 \le k \le |S|$. That is, the function $\mathrm{e}\,(S,k)$ returns the kth element of S.

Definition 10 (Concatenation of ordered sets) *Given any two ordered sets,*

$$S = \left[s_1, s_2, \dots, s_k, \dots, s_{|S|}\right]$$

and

$$T = \left[t_1, t_2, \dots, t_k, \dots, t_{|T|}\right]$$

then, by definition,

$$S \oplus T = \left[s_1, s_2, \dots, s_k, \dots, s_{|S|}, t_1, t_2, \dots, t_k, \dots, t_{|T|}\right]$$

$S \oplus T$ is called the concatenation of T onto S*.*

Theorem 11 (Associativity of ordered set concatenation) *The concatenation operation on ordered sets is* associative*. That is, if R, S and T are ordered sets then*

$$R \oplus (S \oplus T) = (R \oplus S) \oplus T$$

The expressions $R \oplus (S \oplus T)$ and $(R \oplus S) \oplus T$ can therefore both be written

$$R \oplus S \oplus T$$

Proof

R1	Let		$R = \left[r_1, r_2, \dots r_{\vert R\vert}\right]$ $S = \left[s_1, s_2, \dots s_{\vert S\vert}\right]$ $T = \left[t_1, t_2, \dots t_{\vert T\vert}\right]$
R2	10 & R1	$\Rightarrow$	$R \oplus (S \oplus T) = R \oplus \left[s_1, s_2, \dots s_{\vert S\vert}, t_1, t_2, \dots t_{\vert T\vert}\right]$
			$= \left[r_1, r_2, \dots r_{\vert R\vert}, s_1, s_2, \dots s_{\vert S\vert}, t_1, t_2, \dots t_{\vert T\vert}\right]$
R3	10 & R1	$\Rightarrow$	$(R \oplus S) \oplus T = \left[r_1, r_2, \dots r_{\vert R\vert}, s_1, s_2, \dots s_{\vert S\vert}\right] \oplus T$
			$= \left[r_1, r_2, \dots r_{\vert R\vert}, s_1, s_2, \dots s_{\vert S\vert}, t_1, t_2, \dots t_{\vert T\vert}\right]$
R4	R2 & R3	$\Rightarrow$	$R \oplus (S \oplus T) = (R \oplus S) \oplus T$

Definition 12 *If $S_1, S_2, \dots S_k, \dots S_n$ is a collection of ordered sets then, by definition,*

$$S_1 \oplus S_2 \oplus \dots \oplus S_k \oplus \dots \oplus S_n = \bigoplus_{k=1}^{n} S_k$$

Definition 13 (Rotation of ordered sets) *Given an ordered set,*

$$S = \left[s_1, s_2, \ldots, s_k, \ldots, s_{|S|}\right]$$

and given that n is a natural number that satisfies the condition

$$0 < n < |S|$$

then, by definition,

$$\rho_0\left(S\right) = S$$

and

$$\rho_n\left(S\right) = \left[s_{n+1}, s_{n+2}, \ldots, s_{|S|}\right] \oplus \left[s_1, s_2, \ldots, s_n\right]$$

Definition 14 (Ordered set equality) *If S and T are two ordered sets,*

$$S = \left[s_1, s_2, \ldots s_{|S|}\right] \qquad T = \left[t_1, t_2, \ldots t_{|T|}\right]$$

then $S = T$ if and only if $|S| = |T|$ and $\mathrm{e}\left(S, k\right) = \mathrm{e}\left(T, k\right)$ for all integer values of k such that $1 \leq k \leq |S|$.

4.1.3 Operations on sets

Definition 15 (Set equality) *If S and T are two sets then S is equal to T, denoted $S = T$, if and only if one of the following two conditions is satisfied:*

1. *Both S and T are equal to the empty set.*

2. *Every element in S is an element in T and every element in T is an element in S.*

If S is not equal to T then this is denoted $S \neq T$.

Definition 16 (Subset) *If S and T are two sets then S is a* subset *of T, denoted $S \subseteq T$, if and only if one of the following two conditions is satisfied:*

1. *S is the empty set.*

2. *Every element of S is also an element of T.*

If S is not a subset of T then this is denoted $S \nsubseteq T$.

Definition 17 (Superset) *If S and T are two sets then S is a* superset *of T, denoted $S \supseteq T$, if and only if one of the following two conditions is satisfied:*

1. *T is the empty set.*

2. *Every element of T is also an element of S.*

If S is not a superset of T then this is denoted $S \nsupseteq T$.

Definition 18 (Proper subset) *If S and T are two sets then S is a* proper subset *of T, denoted $S \subset T$, if and only if every element of S is also an element of T, S is not the empty set and $S \neq T$. If S is not a proper subset of T then this is denoted $S \not\subset T$.*

Definition 19 (Proper superset) *If S and T are two sets then S is a* proper superset *of T, denoted $S \supset T$, if and only if every element of T is also an element of S, T is not the empty set and $S \neq T$. If S is not a proper superset of T then this is denoted $S \not\supset T$.*

Definition 20 (Set union) *If S and T are two sets then the* union *of S and T, denoted $S \cup T$, is the set that only contains every object that is an element of S or an element of T or an element of both S and T. That is*

$$(s \in (S \cup T)) \iff ((s \in S) \vee (s \in T))$$

Theorem 21 (Associativity of set union) *The union operation on sets is* associative*. That is, if R, S and T are sets then*

$$R \cup (S \cup T) = (R \cup S) \cup T$$

The expressions $R \cup (S \cup T)$ and $(R \cup S) \cup T$ can therefore both be written

$$R \cup S \cup T$$

Proof

R1	Let		R, S and T be sets.
R2	R1 & 20	$\Rightarrow$	$(v \in (R \cup S)) \iff ((v \in R) \vee (v \in S))$
R3	R1 & 20	$\Rightarrow$	$(v \in ((R \cup S) \cup T)) \iff ((v \in (R \cup S)) \vee (v \in T))$
R4	R2 & R3	$\Rightarrow$	$(v \in ((R \cup S) \cup T)) \iff ((v \in R) \vee (v \in S) \vee (v \in T))$
R5	R1 & 20	$\Rightarrow$	$(v \in (S \cup T)) \iff ((v \in S) \vee (v \in T))$
R6	R1 & 20	$\Rightarrow$	$(v \in (R \cup (S \cup T))) \iff ((v \in R) \vee (v \in (S \cup T)))$
R7	R5 & R6	$\Rightarrow$	$(v \in (R \cup (S \cup T))) \iff ((v \in R) \vee (v \in S) \vee (v \in T))$
R8	R4 & R7	$\Rightarrow$	$(v \in ((R \cup S) \cup T)) \iff (v \in (R \cup (S \cup T)))$
R9	R8	$\Rightarrow$	$(R \cup S) \cup T = R \cup (S \cup T)$

Definition 22 (Union of sequence of sets) *If $S_1, S_2, \ldots S_k, \ldots S_n$ is a collection of sets then, by definition,*

$$S_1 \cup S_2 \cup \ldots \cup S_k \cup \ldots \cup S_n = \bigcup_{k=1}^{n} S_k$$

Also, if S is a set, then

$$\bigcup_{s \in S} \mathrm{F}(s)$$

returns the set that contains all and only those objects that are members of one or more of the sets $\mathrm{F}(s)$ where s only takes any value such that $s \in S$ and where $\mathrm{F}(s)$ is some function of s that returns a set.

Definition 23 (Set intersection) *If S and T are two sets then the* intersection *of S and T, denoted $S \cap T$, is the set that only contains every object s that is a member of S and a member of T:*

$$(s \in (S \cap T)) \iff ((s \in S) \wedge (s \in T))$$

Theorem 24 *The intersection operation on sets is* associative. *That is, if R, S and T are sets then*

$$R \cap (S \cap T) = (R \cap S) \cap T$$

The expressions $R \cap (S \cap T)$ and $(R \cap S) \cap T$ can therefore both be written

$$R \cap S \cap T$$

Proof

R1	Let		R, S and T be sets.
R2	R1 & 23	$\Rightarrow$	$(v \in (R \cap S)) \iff ((v \in R) \wedge (v \in S))$
R3	R1 & 23	$\Rightarrow$	$(v \in ((R \cap S) \cap T)) \iff ((v \in (R \cap S)) \wedge (v \in T))$
R4	R2 & R3	$\Rightarrow$	$(v \in ((R \cap S) \cap T)) \iff ((v \in R) \wedge (v \in S) \wedge (v \in T))$
R5	R1 & 23	$\Rightarrow$	$(v \in (S \cap T)) \iff ((v \in S) \wedge (v \in T))$
R6	R1 & 23	$\Rightarrow$	$(v \in (R \cap (S \cap T))) \iff ((v \in R) \wedge (v \in (S \cap T)))$
R7	R5 & R6	$\Rightarrow$	$(v \in (R \cap (S \cap T))) \iff ((v \in R) \wedge (v \in S) \wedge (v \in T))$
R8	R4 & R7	$\Rightarrow$	$(v \in ((R \cap S) \cap T)) = (v \in (R \cap (S \cap T)))$

Definition 25 *If $S_1, S_2, \ldots S_k, \ldots S_n$ is a collection of sets then, by definition,*

$$S_1 \cap S_2 \cap \ldots \cap S_k \cap \ldots \cap S_n = \bigcap_{k=1}^{n} S_k$$

Definition 26 (Set partition) *If S is a set then $\mathrm{P}(S)$ is a* partition on S *if and only if the following conditions are satisfied:*

1. $\mathrm{P}(S)$ *is a set.*
2. $\bigcup_{s \in \mathrm{P}(S)} s = S$.
3. $(s_1, s_2 \in \mathrm{P}(S)) \wedge (s_1 \neq s_2) \Rightarrow (s_1 \cap s_2 = \emptyset)$.

4.2 Arithmetic

4.2.1 int

Definition 27 (int) *The function* $\mathrm{int}(x)$ *takes any real number x as its argument and returns the largest integer less than or equal to x. In other words,* $\mathrm{int}(x)$ *is defined as follows:*

$$\mathrm{int}(x) = y : (x - 1 < y \leq x) \wedge (y \in \mathbb{Z})$$

where $\mathbb{Z}$ is the universal set of integers.

Theorem 28 *For any pair of real numbers a and b,*

$$\mathrm{int}\,(a - \mathrm{int}\,(b)) = \mathrm{int}\,(a) - \mathrm{int}\,(b)$$

Proof

R1	27	$\Rightarrow$	$a - \mathrm{int}\,(b) - 1 < \mathrm{int}\,(a - \mathrm{int}\,(b)) \leq a - \mathrm{int}\,(b)$
R2	27	$\Rightarrow$	$a - 1 < \mathrm{int}\,(a) \leq a$
R3	R2	$\Rightarrow$	$a - 1 - \mathrm{int}\,(b) < \mathrm{int}\,(a) - \mathrm{int}\,(b) \leq a - \mathrm{int}\,(b)$
R4	27	$\Rightarrow$	$\mathrm{int}\,(a - \mathrm{int}\,(b)) \in \mathbb{Z}$ and $(\mathrm{int}\,(a) - \mathrm{int}\,(b)) \in \mathbb{Z}$
R5	R1, R3 & R4	$\Rightarrow$	$\mathrm{int}\,(a - \mathrm{int}\,(b)) = \mathrm{int}\,(a) - \mathrm{int}\,(b)$

Theorem 29 *For any pair of real numbers a and b,*

$$\mathrm{int}\,(a + \mathrm{int}\,(b)) = \mathrm{int}\,(a) + \mathrm{int}\,(b)$$

Proof

R1	27	$\Rightarrow$	$a + \mathrm{int}\,(b) - 1 < \mathrm{int}\,(a + \mathrm{int}\,(b)) \leq a + \mathrm{int}\,(b)$
R2	27	$\Rightarrow$	$a - 1 < \mathrm{int}\,(a) \leq a$
R3	R2	$\Rightarrow$	$a - 1 + \mathrm{int}\,(b) < \mathrm{int}\,(a) + \mathrm{int}\,(b) \leq a + \mathrm{int}\,(b)$
R4	27	$\Rightarrow$	$\mathrm{int}\,(a + \mathrm{int}\,(b)) \in \mathbb{Z}$ and $(\mathrm{int}\,(a) + \mathrm{int}\,(b)) \in \mathbb{Z}$
R5	R1, R3 & R4	$\Rightarrow$	$\mathrm{int}\,(a + \mathrm{int}\,(b)) = \mathrm{int}\,(a) + \mathrm{int}\,(b)$

Theorem 30 *For any pair of real numbers a and b,*

$$\mathrm{int}\,(a + b) = \mathrm{int}\,(a) + \mathrm{int}\,(b) + \mathrm{int}\,(a + b - \mathrm{int}\,(a) - \mathrm{int}\,(b))$$

Proof

R1 29 $\Rightarrow$
$$\begin{aligned} &\mathrm{int}\,(a) + \mathrm{int}\,(b) + \mathrm{int}\,(a + b - \mathrm{int}\,(a) - \mathrm{int}\,(b)) \\ &\quad = \mathrm{int}\,(a) + \mathrm{int}\,(b) + \mathrm{int}\,(a + b - (\mathrm{int}\,(a) + \mathrm{int}\,(b))) \\ &\quad = \mathrm{int}\,(a + \mathrm{int}\,(b)) + \mathrm{int}\,(a + b - \mathrm{int}\,(a + \mathrm{int}\,(b))) \end{aligned}$$

R2 R1 & 28 $\Rightarrow$
$$\begin{aligned} &\mathrm{int}\,(a) + \mathrm{int}\,(b) + \mathrm{int}\,(a + b - \mathrm{int}\,(a) - \mathrm{int}\,(b)) \\ &\quad = \mathrm{int}\,(a + \mathrm{int}\,(b)) + \mathrm{int}\,(a + b) - \mathrm{int}\,(a + \mathrm{int}\,(b)) \\ &\quad = \mathrm{int}\,(a + b) \end{aligned}$$

Theorem 31 *For any pair of real numbers a and b,*

$$\mathrm{int}\,(a - b) = \mathrm{int}\,(a) - \mathrm{int}\,(b) + \mathrm{int}\,(a - b - \mathrm{int}\,(a) + \mathrm{int}\,(b))$$

Proof

$$\begin{array}{lll} \text{R1} & 28 & \Rightarrow \quad \text{int}(a) - \text{int}(b) + \text{int}(a - b - \text{int}(a) + \text{int}(b)) \\ & & \quad = \text{int}(a - \text{int}(b)) + \text{int}(a - b - \text{int}(a - \text{int}(b))) \\ & & \quad = \text{int}(a - \text{int}(b)) + \text{int}(a - b) - \text{int}(a - \text{int}(b)) \\ & & \quad = \text{int}(a - b) \end{array}$$

Theorem 32 *Given any two real numbers, a and c; an integer, b; and a non-zero real number y then*

$$\text{int}(a + b \times \text{int}(c)) = \text{int}(a) + b \times \text{int}(c)$$

Proof

$$\begin{array}{llll} \text{R1} & \text{Let} & & b \in \mathbb{Z} \\ \text{R2} & 27 & \Rightarrow & (a + b \times \text{int}(c) - 1 < \text{int}(a + b \times \text{int}(c)) \leq a + b \times \text{int}(c)) \wedge (\text{int}(a + b \times \text{int}(c)) \in \mathbb{Z}) \\ \text{R3} & \text{R1 \& 27} & \Rightarrow & (b \times \text{int}(c)) \in \mathbb{Z} \\ \text{R4} & 27 & \Rightarrow & (a - 1 < \text{int}(a) \leq a) \wedge (\text{int}(a) \in \mathbb{Z}) \\ \text{R5} & \text{R3 \& R4} & \Rightarrow & (a - 1 + b \times \text{int}(c) < \text{int}(a) + b \times \text{int}(c) \leq a + b \times \text{int}(c)) \wedge ((\text{int}(a) + b \times \text{int}(c)) \in \mathbb{Z}) \\ \text{R6} & \text{R2 \& R5} & \Rightarrow & \text{int}(a + b \times \text{int}(c)) = \text{int}(a) + b \times \text{int}(c) \end{array}$$

4.2.2 mod

Definition 33 (mod) *Given that x is a real number and y is a non-zero real number, then the binary operation* mod *is defined as follows:*

$$x \bmod y = x - y \times \text{int}\left(\frac{x}{y}\right)$$

Theorem 34 *For any pair of real numbers a and b and any non-zero real number y,*

$$(a + b) \bmod y = (a \bmod y + b \bmod y) \bmod y$$

Proof

R1	33	$\Rightarrow$	$(a+b) \bmod y = (a+b) - y \times \text{int}\left(\frac{a+b}{y}\right)$
R2	33	$\Rightarrow$	$(a \bmod y + b \bmod y) \bmod y$ $= \left(a - y \times \text{int}\left(\frac{a}{y}\right) + b - y \times \text{int}\left(\frac{b}{y}\right)\right)$ $- y \times \text{int}\left(\frac{\left(a-y\times\text{int}\left(\frac{a}{y}\right)+b-y\times\text{int}\left(\frac{b}{y}\right)\right)}{y}\right)$
R3	R2	$\Rightarrow$	$(a \bmod y + b \bmod y) \bmod y$ $= a + b - y \times \left(\text{int}\left(\frac{a}{y}\right) + \text{int}\left(\frac{b}{y}\right) + \text{int}\left(\frac{\left(a-y\times\text{int}\left(\frac{a}{y}\right)+b-y\times\text{int}\left(\frac{b}{y}\right)\right)}{y}\right)\right)$ $= a + b - y \times \left(\text{int}\left(\frac{a}{y}\right) + \text{int}\left(\frac{b}{y}\right) + \text{int}\left(\frac{a}{y} - \text{int}\left(\frac{a}{y}\right) + \frac{b}{y} - \text{int}\left(\frac{b}{y}\right)\right)\right)$
R4	30	$\Rightarrow$	$\text{int}\left(\frac{a}{y} - \text{int}\left(\frac{a}{y}\right) + \frac{b}{y} - \text{int}\left(\frac{b}{y}\right)\right) = \text{int}\left(\frac{a}{y} + \frac{b}{y}\right) - \text{int}\left(\frac{a}{y}\right) - \text{int}\left(\frac{b}{y}\right)$
R5	R3 & R4	$\Rightarrow$	$(a \bmod y + b \bmod y) \bmod y$ $= a + b - y \times \left(\text{int}\left(\frac{a}{y}\right) + \text{int}\left(\frac{b}{y}\right) + \text{int}\left(\frac{a}{y} + \frac{b}{y}\right) - \text{int}\left(\frac{a}{y}\right) - \text{int}\left(\frac{b}{y}\right)\right)$ $= (a+b) - y \times \text{int}\left(\frac{a}{y} + \frac{b}{y}\right)$ $= (a+b) - y \times \text{int}\left(\frac{a+b}{y}\right)$
R6	R1 & R5	$\Rightarrow$	$(a \bmod y + b \bmod y) \bmod y = (a+b) \bmod y$

Theorem 35 *For any real number a and any non-zero real number y,*

$$(a \bmod y) \bmod y = a \bmod y$$

Proof

R1	33	$\Rightarrow$	$a \bmod y = a - y \times \text{int}\left(\frac{a}{y}\right)$
R2	33	$\Rightarrow$	$(a \bmod y) \bmod y = a - y \times \text{int}\left(\frac{a}{y}\right) - y \times \text{int}\left(\frac{a-y\times\text{int}(a/y)}{y}\right)$
R3	R2	$\Rightarrow$	$(a \bmod y) \bmod y = a - y \times \text{int}\left(\frac{a}{y}\right) - y \times \text{int}\left(\frac{a}{y} - \text{int}\left(\frac{a}{y}\right)\right)$
R4	R3 & 28	$\Rightarrow$	$(a \bmod y) \bmod y$ $= a - y \times \text{int}\left(\frac{a}{y}\right) - y \times \left(\text{int}\left(\frac{a}{y}\right) - \text{int}\left(\frac{a}{y}\right)\right)$ $= a - y \times \text{int}\left(\frac{a}{y}\right)$
R5	R1 & R4	$\Rightarrow$	$(a \bmod y) \bmod y = a \bmod y$

Theorem 36 *For any integer b and any non-zero real number y,*

$$by \bmod y = 0$$

Proof

R1	33	$\Rightarrow$	$by \bmod y = by - y \times \operatorname{int}\left(\frac{by}{y}\right)$ $= by - y \times \operatorname{int}(b)$
R2	Let		$b \in \mathbb{Z}$
R3	R2 & 27	$\Rightarrow$	$\operatorname{int}(b) = b$
R4	R1 & R3	$\Rightarrow$	$by \bmod y = by - y \times b = 0$

Theorem 37 *For any real number a, any integer b and any non-zero real number y,*

$$(a + by) \bmod y = a \bmod y$$

Proof

R1	34	$\Rightarrow$	$(a + by) \bmod y = (a \bmod y + by \bmod y) \bmod y$
R2	36	$\Rightarrow$	$by \bmod y = 0$
R3	R1 & R2	$\Rightarrow$	$(a + by) \bmod y = (a \bmod y) \bmod y$
R4	R3 & 35	$\Rightarrow$	$(a + by) \bmod y = a \bmod y$

Theorem 38 *For any pair of real numbers a and b and any non-zero real number y,*

$$(a \bmod y + b) \bmod y = (a + b) \bmod y$$

Proof

R1	33	$\Rightarrow$	$(a+b) \bmod y = (a+b) - y \times \mathrm{int}\left(\frac{a+b}{y}\right)$
R2	33	$\Rightarrow$	$(a \bmod y + b) \bmod y$ $= \left(a - y \times \mathrm{int}\left(\frac{a}{y}\right) + b\right) - y \times \mathrm{int}\left(\frac{a - y \times \mathrm{int}\left(\frac{a}{y}\right) + b}{y}\right)$
R3	R2	$\Rightarrow$	$(a \bmod y + b) \bmod y$ $= a + b - y \times \left(\mathrm{int}\left(\frac{a}{y}\right) + \mathrm{int}\left(\frac{a - y \times \mathrm{int}(a/y) + b}{y}\right)\right)$ $= a + b - y \times \left(\mathrm{int}\left(\frac{a}{y}\right) + \mathrm{int}\left(\frac{a}{y} - \mathrm{int}\left(\frac{a}{y}\right) + \frac{b}{y}\right)\right)$
R4	28	$\Rightarrow$	$\mathrm{int}\left(\frac{a}{y} - \mathrm{int}\left(\frac{a}{y}\right) + \frac{b}{y}\right) = \mathrm{int}\left(\frac{a}{y} + \frac{b}{y}\right) - \mathrm{int}\left(\frac{a}{y}\right)$
R5	R3 & R4	$\Rightarrow$	$(a \bmod y + b) \bmod y$ $= a + b - y \times \left(\mathrm{int}\left(\frac{a}{y}\right) + \mathrm{int}\left(\frac{a}{y} + \frac{b}{y}\right) - \mathrm{int}\left(\frac{a}{y}\right)\right)$ $= (a+b) - y \times \mathrm{int}\left(\frac{a}{y} + \frac{b}{y}\right)$
R6	R1 & R5	$\Rightarrow$	$(a \bmod y + b) \bmod y = (a+b) \bmod y$

Theorem 39 *Given a real number b, a collection of real numbers $a_1, a_2, \ldots a_k$ and a non-zero real number y,*

$$\left(\sum_{j=1}^{k} ((a_j \times b) \bmod y)\right) \bmod y = \left(\left(\sum_{j=1}^{k} a_j\right) \times b\right) \bmod y$$

Proof

R1 33 $\Rightarrow$ $\left(\sum_{j=1}^{k}\left((a_j b) \bmod y\right)\right) \bmod y$

$$= \left(\sum_{j=1}^{k}\left((a_j b) - y \times \mathrm{int}\left(\frac{a_j b}{y}\right)\right)\right)$$

$$-y \times \mathrm{int}\left(\frac{\sum_{j=1}^{k}\left((a_j b) - y \times \mathrm{int}\left(\frac{a_j b}{y}\right)\right)}{y}\right)$$

$$= \left(\sum_{j=1}^{k}(a_j b)\right) - y \times \left(\sum_{j=1}^{k}\left(\mathrm{int}\left(\frac{a_j b}{y}\right)\right)\right)$$

$$-y \times \mathrm{int}\left(\frac{\sum_{j=1}^{k}(a_j b)}{y} - \left(\sum_{j=1}^{k}\left(\mathrm{int}\left(\frac{a_j b}{y}\right)\right)\right)\right)$$

$$= \left(\sum_{j=1}^{k}(a_j b)\right)$$

$$-y \times \left(\begin{array}{l} \left(\sum_{j=1}^{k}\left(\mathrm{int}\left(\frac{a_j b}{y}\right)\right)\right) \\ + \mathrm{int}\left(\frac{\sum_{j=1}^{k}(a_j b)}{y} - \left(\sum_{j=1}^{k}\left(\mathrm{int}\left(\frac{a_j b}{y}\right)\right)\right)\right) \end{array}\right)$$

$$= \left(\sum_{j=1}^{k}(a_j b)\right)$$

$$-y \times \left(\begin{array}{l} \mathrm{int}\left(\sum_{j=1}^{k}\left(\mathrm{int}\left(\frac{a_j b}{y}\right)\right)\right) \\ + \mathrm{int}\left(\frac{\sum_{j=1}^{k}(a_j b)}{y} - \mathrm{int}\left(\sum_{j=1}^{k}\left(\mathrm{int}\left(\frac{a_j b}{y}\right)\right)\right)\right) \end{array}\right)$$

R2 R1 & 28 $\Rightarrow$ $\left(\sum_{j=1}^{k}\left((a_j b) \bmod y\right)\right) \bmod y$

$$= \left(\sum_{j=1}^{k}(a_j b)\right)$$

$$-y \times \left(\begin{array}{l} \mathrm{int}\left(\sum_{j=1}^{k}\left(\mathrm{int}\left(\frac{a_j b}{y}\right)\right)\right) \\ + \mathrm{int}\left(\frac{\sum_{j=1}^{k}(a_j b)}{y}\right) \\ - \mathrm{int}\left(\sum_{j=1}^{k}\left(\mathrm{int}\left(\frac{a_j b}{y}\right)\right)\right) \end{array}\right)$$

$$= \left(\sum_{j=1}^{k}(a_j b)\right) - y \times \mathrm{int}\left(\frac{\sum_{j=1}^{k}(a_j b)}{y}\right)$$

R3 R2 & 33 $\Rightarrow$ $\left(\sum_{j=1}^{k}\left((a_j b) \bmod y\right)\right) \bmod y = \left(\sum_{j=1}^{k}(a_j b)\right) \bmod y$

Theorem 40 *Given any three real numbers a, b and c and a non-zero real number y,*

$$((a+b) \bmod y = (a+c) \bmod y) \iff \left(\frac{c-b}{y} \in \mathbb{Z}\right)$$

where $\mathbb{Z}$ is the universal set of integers.

Proof

R1	33	$\Rightarrow$	$(a+b) \bmod y = (a+b) - y \times \mathrm{int}\left(\frac{a+b}{y}\right)$
R2	33	$\Rightarrow$	$(a+c) \bmod y = (a+c) - y \times \mathrm{int}\left(\frac{a+c}{y}\right)$
R3	Let		$(a+b) \bmod y = (a+c) \bmod y$
R4	R1, R2, R3 & 27	$\Rightarrow$	$(a+b) - y \times \mathrm{int}\left(\frac{a+b}{y}\right) = (a+c) - y \times \mathrm{int}\left(\frac{a+c}{y}\right)$
			$\Rightarrow \quad b = c - y \times \mathrm{int}\left(\frac{a+c}{y}\right) + y \times \mathrm{int}\left(\frac{a+b}{y}\right)$
			$\Rightarrow \quad b = c - y \times \left(\mathrm{int}\left(\frac{a+c}{y}\right) - \mathrm{int}\left(\frac{a+b}{y}\right)\right)$
			$\Rightarrow \quad \frac{c-b}{y} = \mathrm{int}\left(\frac{a+c}{y}\right) - \mathrm{int}\left(\frac{a+b}{y}\right)$
			$\Rightarrow \quad \frac{c-b}{y} \in \mathbb{Z}$
R5	R1 to R4	$\Rightarrow$	$((a+b) \bmod y = (a+c) \bmod y) \Rightarrow \left(\frac{c-b}{y} \in \mathbf{Z}\right)$
R6	Let		$\frac{c-b}{y} = n$ where $n \in \mathbb{Z}$
R7	R6	$\Rightarrow$	$c = n \times y + b$
R8	R7	$\Rightarrow$	$(a+c) \bmod y = (a+b+n \times y) \bmod y$
R9	R8 & 37	$\Rightarrow$	$(a+c) \bmod y = (a+b) \bmod y$
R10	R6 to R9	$\Rightarrow$	$\left(\frac{c-b}{y} \in \mathbf{Z}\right) \Rightarrow ((a+b) \bmod y = (a+c) \bmod y)$
R11	R5 & R10	$\Rightarrow$	$((a+b) \bmod y = (a+c) \bmod y) \iff \left(\frac{c-b}{y} \in \mathbb{Z}\right)$

Theorem 41 *Given any real number a and any non-zero real number y,*

$$(y > 0) \Rightarrow (y > a \bmod y \geq 0)$$

Proof

R1	Let		$y > 0$
R2	33	$\Rightarrow$	$a \bmod y = a - y \times \text{int}\left(\frac{a}{y}\right)$
R3	27	$\Rightarrow$	$\frac{a}{y} - 1 < \text{int}\left(\frac{a}{y}\right) \leq \frac{a}{y}$
R4	R1 & R3	$\Rightarrow$	$a - y < y \times \text{int}\left(\frac{a}{y}\right) \leq a$
R5	R4	$\Rightarrow$	$y - a > -y \times \text{int}\left(\frac{a}{y}\right) \geq -a$
R6	R5	$\Rightarrow$	$y > a - y \times \text{int}\left(\frac{a}{y}\right) \geq 0$
R7	R2 & R6	$\Rightarrow$	$y > a \bmod y \geq 0$
R8	R1 to R7	$\Rightarrow$	$(y > 0) \Rightarrow (y > a \bmod y \geq 0)$

Theorem 42 *Given any real number a and any non-zero real number y,*

$$(y < 0) \Rightarrow (y < a \bmod y \leq 0)$$

Proof

R1	Let		$y < 0$
R2	33	$\Rightarrow$	$a \bmod y = a - y \times \text{int}\left(\frac{a}{y}\right)$
R3	27	$\Rightarrow$	$\frac{a}{y} - 1 < \text{int}\left(\frac{a}{y}\right) \leq \frac{a}{y}$
R4	R1 & R3	$\Rightarrow$	$a - y > y \times \text{int}\left(\frac{a}{y}\right) \geq a$
R5	R4	$\Rightarrow$	$y - a < -y \times \text{int}\left(\frac{a}{y}\right) \leq -a$
R6	R5	$\Rightarrow$	$y < a - y \times \text{int}\left(\frac{a}{y}\right) \leq 0$
R7	R2 & R6	$\Rightarrow$	$y < a \bmod y \leq 0$
R8	R1 to R7	$\Rightarrow$	$(y < 0) \Rightarrow (y < a \bmod y \leq 0)$

Theorem 43 *If a, b, c and y are real numbers then*

$$(y > a, b, c \geq 0) \wedge (a = (b - c) \bmod y) \Rightarrow (b = (a + c) \bmod y)$$

Proof

R1	Let		$y > a, b, c \geq 0$
R2	Let		$a = (b - c) \bmod y$
R3	R2 & 33	$\Rightarrow$	$a = b - c - y \times \text{int}\left(\frac{b-c}{y}\right)$
R4	R1 & 27	$\Rightarrow$	$c > b \Rightarrow \text{int}\left(\frac{b-c}{y}\right) = -1$
R5	R1 & 27	$\Rightarrow$	$c \leq b \Rightarrow \text{int}\left(\frac{b-c}{y}\right) = 0$
R6	R3 & R4	$\Rightarrow$	$c > b \Rightarrow a = b - c + y \Rightarrow a + c = b + y$
R7	R1 & R6	$\Rightarrow$	$c > b \Rightarrow a + c \geq y$
R8	R3 & R5	$\Rightarrow$	$c \leq b \Rightarrow a = b - c \Rightarrow a + c = b$
R9	R1 & R8	$\Rightarrow$	$c \leq b \Rightarrow a + c < y$
R10	R9	$\Rightarrow$	$a + c \geq y \Rightarrow c \nleq b \Rightarrow c > b$
R11	R7	$\Rightarrow$	$a + c < y \Rightarrow c \ngtr b \Rightarrow c \leq b$
R12	R6 & R10	$\Rightarrow$	$a + c \geq y \Rightarrow b = a + c - y$
R13	R8 & R11	$\Rightarrow$	$a + c < y \Rightarrow b = a + c$
R14	R12 & R13	$\Rightarrow$	$b = \begin{cases} a + c - y & \text{if} \quad a + c \geq y \\ a + c & \text{if} \quad a + c < y \end{cases}$
R15	Let		$z = (a + c) \bmod y$
R16	R15 & 33	$\Rightarrow$	$z = a + c - y \times \text{int}\left(\frac{a+c}{y}\right)$
R17	R1 & 27	$\Rightarrow$	$a + c \geq y \Rightarrow \text{int}\left(\frac{a+c}{y}\right) = 1$
R18	R1 & 27	$\Rightarrow$	$a + c < y \Rightarrow \text{int}\left(\frac{a+c}{y}\right) = 0$
R19	R16 & R17	$\Rightarrow$	$a + c \geq y \Rightarrow z = a + c - y$

R20	R16 & R18	$\Rightarrow$	$a + c < y \Rightarrow z = a + c$
R21	R19 & R20	$\Rightarrow$	$z = \begin{cases} a + c - y & \text{if} \quad a + c \geq y \\ a + c & \text{if} \quad a + c < y \end{cases}$
R22	R14 & R21	$\Rightarrow$	$b = z$
R23	R15 & R22	$\Rightarrow$	$b = (a + c) \bmod y$
R24	R1, R2 & R23	$\Rightarrow$	$\left.\begin{array}{l} y > a, b, c \geq 0 \\ a = (b - c) \bmod y \end{array}\right\} \Rightarrow b = (a + c) \bmod y$

Theorem 44 *If a and y are real numbers then*

$$(y > a \geq 0) \Rightarrow (a \bmod y = a)$$

Proof

R1	Let		$y > a \geq 0$
R2	33	$\Rightarrow$	$a \bmod y = a - y \times \operatorname{int}(a/y)$
R3	R1 & 27	$\Rightarrow$	$\operatorname{int}(a/y) = 0$
R4	R2 & R3	$\Rightarrow$	$a \bmod y = a$
R5	R1 to R4	$\Rightarrow$	$(y > a \geq 0) \Rightarrow (a \bmod y = a)$

Theorem 45 *For any real number a, any integer b and any non-zero real number y*

$$(a \times (b \bmod y)) \bmod y = (ab) \bmod y$$

Proof

R1	33	$\Rightarrow$	$(a \times (b \bmod y)) \bmod y$
			$= a \times (b \bmod y) - y \times \mathrm{int}\left(\frac{a \times (b \bmod y)}{y}\right)$
			$= a \times \left(b - y \times \mathrm{int}\left(\frac{b}{y}\right)\right) - y \times \mathrm{int}\left(\frac{a \times \left(b - y \times \mathrm{int}\left(\frac{b}{y}\right)\right)}{y}\right)$
			$= ab - ay \times \mathrm{int}\left(\frac{b}{y}\right) - y \times \mathrm{int}\left(\frac{ab}{y} - a \times \mathrm{int}\left(\frac{b}{y}\right)\right)$
R2	R1 & 32	$\Rightarrow$	$(a \times (b \bmod y)) \bmod y$
			$= ab - ay \times \mathrm{int}\left(\frac{b}{y}\right) - y \times \left(\mathrm{int}\left(\frac{ab}{y}\right) - a \times \mathrm{int}\left(\frac{b}{y}\right)\right)$
			$= ab - ay \times \mathrm{int}\left(\frac{b}{y}\right) - y \times \mathrm{int}\left(\frac{ab}{y}\right) + ay \times \mathrm{int}\left(\frac{b}{y}\right)$
			$= ab - y \times \mathrm{int}\left(\frac{ab}{y}\right)$
R3	R2 & 33	$\Rightarrow$	$(a \times (b \bmod y)) \bmod y = (ab) \bmod y$

Theorem 46 *For any non-zero real number y and any real number a such that $0 \leq a < y$,*

$$a + (-a) \bmod y = y$$

Proof

R1	Let		$0 \leq a < y$
R2	33	$\Rightarrow$	$(-a) \bmod y = -a - y \times \mathrm{int}\left(\frac{-a}{y}\right)$
R3	R1	$\Rightarrow$	$\mathrm{int}\left(\frac{-a}{y}\right) = -1$
R4	R2 & R3	$\Rightarrow$	$(-a) \bmod y = -a - y \times (-1) = -a + y = y - a$
R5	R4	$\Rightarrow$	$(-a) \bmod y = a + y - a = y$

Theorem 47 *For any non-zero real number y, any pair of real numbers x_1 and x_2, and any pair of integers n_1 and n_2,*

$$(x_1 - yn_1 = x_2 - yn_2) \Rightarrow (x_1 \bmod y = x_2 \bmod y)$$

Proof

R1	Let		$x_1 - yn_1 = x_2 - yn_2$
R2	34 & R1	$\Rightarrow$	$(x_1 - yn_1) \bmod y = (x_2 - yn_2) \bmod y$
		$\Rightarrow$	$(x_1 \bmod y - yn_1 \bmod y) \bmod y = (x_2 \bmod y - yn_2 \bmod y) \bmod y$
R3	36 & R2	$\Rightarrow$	$(x_1 \bmod y - 0) \bmod y = (x_2 \bmod y - 0) \bmod y$
R4	R3 & 35	$\Rightarrow$	$x_1 \bmod y = x_2 \bmod y$
R5	R1 to R4	$\Rightarrow$	$(x_1 - yn_1 = x_2 - yn_2) \Rightarrow (x_1 \bmod y = x_2 \bmod y)$

4.2.3 div

Definition 48 (div) *If x is a real number and y is a non-zero real number then the binary operation* div *is defined as follows:*

$$x \text{ div } y = \text{int}\left(\frac{x}{y}\right)$$

Theorem 49 *For any real number x and any non-zero real number y,*

$$x = x \bmod y + y \times (x \text{ div } y)$$

Proof

R1	33	$\Rightarrow$	$x \bmod y = x - y \times \text{int}\left(\frac{x}{y}\right)$
R2	48	$\Rightarrow$	$x \text{ div } y = \text{int}\left(\frac{x}{y}\right)$
R3	R1 & R2	$\Rightarrow$	$x \bmod y + y \times (x \text{ div } y) = x - y \times \text{int}\left(\frac{x}{y}\right) + y \times \text{int}\left(\frac{x}{y}\right) = x$

Theorem 50 *For any real number a, any non-zero real number y and any integer b,*

$$(a - by) \text{ div } y = (a \text{ div } y) - b$$

Proof

R1	48	$\Rightarrow$	$a \text{ div } y - b = \text{int}\left(\frac{a}{y}\right) - b$
R2	48	$\Rightarrow$	$(a - by) \text{ div } y = \text{int}\left(\frac{a-by}{y}\right) = \text{int}\left(\frac{a}{y} - b\right)$
R3	Let		$b \in \mathbb{Z}$
R4	R3	$\Rightarrow$	$b = \text{int}\,(b)$
R5	R2 & 31	$\Rightarrow$	$(a - by) \text{ div } y = \text{int}\left(\frac{a}{y}\right) - \text{int}\,(b) + \text{int}\left(\frac{a}{y} - b - \text{int}\left(\frac{a}{y}\right) + \text{int}\,(b)\right)$
R6	R4 & R5	$\Rightarrow$	$(a - by) \text{ div } y$
			$= \text{int}\left(\frac{a}{y}\right) - b + \text{int}\left(\frac{a}{y} - b - \text{int}\left(\frac{a}{y}\right) + b\right)$
			$= \text{int}\left(\frac{a}{y}\right) - b + \text{int}\left(\frac{a}{y} - \text{int}\left(\frac{a}{y}\right)\right)$
R7	R6 & 28	$\Rightarrow$	$(a - by) \text{ div } y$
			$= \text{int}\left(\frac{a}{y}\right) - b + \text{int}\left(\frac{a}{y}\right) - \text{int}\left(\frac{a}{y}\right)$
			$= \text{int}\left(\frac{a}{y}\right) - b$
R8	R1 & R7	$\Rightarrow$	$(a - by) \text{ div } y = (a \text{ div } y) - b$

Theorem 51 *For any pair of real numbers a and b and any non-zero real number y,*

$$(a + b) \text{ div } y + ((a + b) \text{ mod } y - a) \text{ div } y = \text{int}\left(\frac{b}{y}\right)$$

Proof

R1 33 & 48 $\Rightarrow$ $(a + b) \text{ div } y + ((a + b) \text{ mod } y - a) \text{ div } y$

$$= \text{int}\left(\frac{a+b}{y}\right) + \text{int}\left(\frac{(a+b) - y \times \text{int}\left(\frac{a+b}{y}\right) - a}{y}\right)$$

$$= \text{int}\left(\frac{a+b}{y}\right) + \text{int}\left(\frac{a}{y} + \frac{b}{y} - \text{int}\left(\frac{a+b}{y}\right) - \frac{a}{y}\right)$$

$$= \text{int}\left(\frac{a+b}{y}\right) + \text{int}\left(\frac{b}{y} - \text{int}\left(\frac{a+b}{y}\right)\right)$$

R2 R1 & 28 $\Rightarrow$ $(a + b) \text{ div } y + ((a + b) \text{ mod } y - a) \text{ div } y$

$$= \text{int}\left(\frac{a+b}{y}\right) + \text{int}\left(\frac{b}{y}\right) - \text{int}\left(\frac{a+b}{y}\right)$$

$$= \text{int}\left(\frac{b}{y}\right)$$

Theorem 52 *For any pair of real numbers a and b and any non-zero real number y,*

$$(a \text{ div } y) + (b + a \text{ mod } y) \text{ div } y = (a + b) \text{ div } y$$

Proof

R1 48 $\Rightarrow$ $(a \text{ div } y) + (b + a \text{ mod } y) \text{ div } y = \text{int}\left(\frac{a}{y}\right) + \text{int}\left(\frac{b + a \text{ mod } y}{y}\right)$

R2 R1 & 33 $\Rightarrow$ $(a \text{ div } y) + (b + a \text{ mod } y) \text{ div } y$

$$= \text{int}\left(\frac{a}{y}\right) + \text{int}\left(\frac{b + (a - y \times \text{int}(a/y))}{y}\right)$$

$$= \text{int}\left(\frac{a}{y}\right) + \text{int}\left(\frac{b}{y} + \frac{a}{y} - \text{int}\left(\frac{a}{y}\right)\right)$$

R3 R2 & 28 $\Rightarrow$ $(a \text{ div } y) + (b + a \text{ mod } y) \text{ div } y$

$$= \text{int}\left(\frac{a}{y}\right) + \text{int}\left(\frac{b}{y} + \frac{a}{y}\right) - \text{int}\left(\frac{a}{y}\right)$$

$$= \text{int}\left(\frac{b}{y} + \frac{a}{y}\right) = \text{int}\left(\frac{a+b}{y}\right)$$

R4 R3 & 48 $\Rightarrow$ $(a \text{ div } y) + (b + a \text{ mod } y) \text{ div } y = (a + b) \text{ div } y$

Theorem 53 *For any real number a and any non-zero real number y,*

$$(a \text{ mod } y) \text{ div } y = 0$$

Proof

R1 33 & 48 $\Rightarrow$ $(a \bmod y) \text{ div } y$

$$= \text{int}\left(\frac{a - y \times \text{int}(a/y)}{y}\right)$$

$$= \text{int}\left(\frac{a}{y} - \text{int}\left(\frac{a}{y}\right)\right)$$

R2 R1 & 28 $\Rightarrow$ $(a \bmod y) \text{ div } y = \text{int}\left(\frac{a}{y}\right) - \text{int}\left(\frac{a}{y}\right) = 0$

Theorem 54 *Given a set of real numbers $a_1, a_2, \ldots, a_k$, a real number b and a non-zero real number y,*

$$\left(\sum_{j=1}^{k} ((a_j b) \text{ div } y)\right) + \left(\left(\sum_{j=1}^{k} ((a_j b) \bmod y)\right) \text{ div } y\right) = \left(b \times \sum_{j=1}^{k} a_j\right) \text{ div } y$$

Proof

R1	48 & 27	$\Rightarrow$	$\sum_{j=1}^{k}((a_j b) \text{ div } y) = \sum_{j=1}^{k}\left(\text{int}\left(\frac{a_j b}{y}\right)\right) = \text{int}\left(\sum_{j=1}^{k}\left(\text{int}\left(\frac{a_j b}{y}\right)\right)\right)$
R2	33 & 48	$\Rightarrow$	$\left(\sum_{j=1}^{k}((a_j b) \text{ mod } y)\right) \text{ div } y$
			$= \text{int}\left(\frac{\sum_{j=1}^{k}((a_j b) - y \times \text{int}((a_j b)/y))}{y}\right)$
			$= \text{int}\left(\frac{\sum_{j=1}^{k}(a_j b) - y \times \sum_{j=1}^{k}(\text{int}((a_j b)/y))}{y}\right)$
			$= \text{int}\left(\frac{\sum_{j=1}^{k}(a_j b)}{y} - \sum_{j=1}^{k}\left(\text{int}\left(\frac{a_j b}{y}\right)\right)\right)$
R3	R1, R2 & 28	$\Rightarrow$	$\left(\sum_{j=1}^{k}((a_j b) \text{ mod } y)\right) \text{ div } y$
			$= \text{int}\left(\frac{\sum_{j=1}^{k}(a_j b)}{y}\right) - \text{int}\left(\sum_{j=1}^{k}\left(\text{int}\left(\frac{a_j b}{y}\right)\right)\right)$
R4	R1 & R3	$\Rightarrow$	$\left(\sum_{j=1}^{k}((a_j b) \text{ div } y)\right) + \left(\left(\sum_{j=1}^{k}((a_j b) \text{ mod } y)\right) \text{ div } y\right)$
			$= \text{int}\left(\sum_{j=1}^{k}\left(\text{int}\left(\frac{a_j b}{y}\right)\right)\right) + \text{int}\left(\frac{\sum_{j=1}^{k}(a_j b)}{y}\right) - \text{int}\left(\sum_{j=1}^{k}\left(\text{int}\left(\frac{a_j b}{y}\right)\right)\right)$
			$= \text{int}\left(\frac{\sum_{j=1}^{k}(a_j b)}{y}\right) = \text{int}\left(\frac{b \times \sum_{j=1}^{k} a_j}{y}\right)$
R5	48	$\Rightarrow$	$\left(b \times \sum_{j=1}^{k} a_j\right) \text{ div } y = \text{int}\left(\frac{b \times \sum_{j=1}^{k} a_j}{y}\right)$
R6	R4 & R5	$\Rightarrow$	$\left(\begin{array}{l}\left(\sum_{j=1}^{k}((a_j b) \text{ div } y)\right) \\ +\left(\left(\sum_{j=1}^{k}((a_j b) \text{ mod } y)\right) \text{ div } y\right)\end{array}\right) = \left(b \times \sum_{j=1}^{k} a_j\right) \text{ div } y$

Theorem 55 *If a and b are any two real numbers and y is any non-zero real number then*

$$(b \text{ div } y) - (a \text{ div } y) + (((b \text{ mod } y) - (a \text{ mod } y)) \text{ div } y) = (b - a) \text{ div } y$$

Proof

R1	Let		$z = (b \operatorname{div} y) - (a \operatorname{div} y) + (((b \bmod y) - (a \bmod y)) \operatorname{div} y)$
R2	R1 & 33	$\Rightarrow$	$z = (b \operatorname{div} y) - (a \operatorname{div} y) + (((b - y \times \operatorname{int}(b/y)) - (a - y \times \operatorname{int}(a/y))) \operatorname{div} y)$
R3	R2 & 48	$\Rightarrow$	$z = \operatorname{int}\left(\frac{b}{y}\right) - \operatorname{int}\left(\frac{b}{y}\right) + \operatorname{int}\left(\frac{b - y \times \operatorname{int}(b/y) - a + y \times \operatorname{int}(a/y)}{y}\right)$
			$= \operatorname{int}\left(\frac{b}{y}\right) - \operatorname{int}\left(\frac{a}{y}\right) + \operatorname{int}\left(\frac{b}{y} - \frac{a}{y} - \operatorname{int}\left(\frac{b}{y}\right) + \operatorname{int}\left(\frac{a}{y}\right)\right)$
R4	R3 & 29	$\Rightarrow$	$z = \operatorname{int}\left(\frac{b}{y}\right) - \operatorname{int}\left(\frac{a}{y}\right) + \operatorname{int}\left(\frac{b}{y} - \frac{a}{y} - \operatorname{int}\left(\frac{b}{y}\right)\right) + \operatorname{int}\left(\frac{a}{y}\right)$
			$= \operatorname{int}\left(\frac{b}{y}\right) + \operatorname{int}\left(\frac{b-a}{y} - \operatorname{int}\left(\frac{b}{y}\right)\right)$
R5	R4 & 28	$\Rightarrow$	$z = \operatorname{int}\left(\frac{b}{y}\right) + \operatorname{int}\left(\frac{b-a}{y}\right) - \operatorname{int}\left(\frac{b}{y}\right)$
			$= \operatorname{int}\left(\frac{b-a}{y}\right)$
R6	48	$\Rightarrow$	$\operatorname{int}\left(\frac{b-a}{y}\right) = (b - a) \operatorname{div} y$
R7	R1, R5 & R6	$\Rightarrow$	$(b \operatorname{div} y) - (a \operatorname{div} y) + (((b \bmod y) - (a \bmod y)) \operatorname{div} y) = (b - a) \operatorname{div} y$

Theorem 56 *If a is an integer and y is a positive, non-zero real number and b is a real number such that $0 \leq b < y$, then*

$$a + (-a \times ((-b) \bmod y)) \operatorname{div} y = (ba) \operatorname{div} y$$

Proof

R1	Let		a be an integer
R2	Let		b be a real number such that $0 \leq b < y$
R3	Let		$z = a + (-a \times ((-b) \text{ mod } y)) \text{ div } y$
R4	R2 & 46	$\Rightarrow$	$(-b) \text{ mod } y = y - b$
R5	R3 & R4	$\Rightarrow$	$z = a + (-a \times (y - b)) \text{ div } y = a + (ba - ay) \text{ div } y$
R6	R5 & 48	$\Rightarrow$	$z = a + \text{int}\left(\frac{ba-ay}{y}\right) = a + \text{int}\left(\frac{ba}{y} - a\right)$
R7	R1	$\Rightarrow$	$a = \text{int}\,(a)$
R8	R6 & R7	$\Rightarrow$	$z = a + \text{int}\left(\frac{ba}{y} - \text{int}\,(a)\right)$
R9	R8 & 28	$\Rightarrow$	$z = a + \text{int}\left(\frac{ba}{y}\right) - \text{int}\,(a)$
R10	R7 & R9	$\Rightarrow$	$z = \text{int}\left(\frac{ba}{y}\right)$
R11	R10 & 48	$\Rightarrow$	$z = (ba) \text{ div } y$
R12	R3 & R11	$\Rightarrow$	$a + (-a \times ((-b) \text{ mod } y)) \text{ div } y = (ba) \text{ div } y$

Theorem 57 *If a is an integer, b is real and y is a non-zero integer then*

$$(ab - a \times (b \text{ mod } y)) \text{ div } y + (a \times (b \text{ mod } y)) \text{ div } y = ab \text{ div } y$$

Proof

R1	Let		a be an integer, b be a real number and y be a non-zero integer
R2	Let		$x = (ab - a \times (b \bmod y)) \text{ div } y + (a \times (b \bmod y)) \text{ div } y$
R3	R2, 48 & 33	$\Rightarrow$	$x = \text{int}\left(\frac{ab - a\times(b - y\times\text{int}(b/y))}{y}\right) + \text{int}\left(\frac{a\times(b-y\times\text{int}(b/y))}{y}\right)$
			$= \text{int}\left(\frac{ab}{y} - \frac{a}{y} \times \left(b - y \times \text{int}\left(\frac{b}{y}\right)\right)\right) + \text{int}\left(\frac{a}{y} \times \left(b - y \times \text{int}\left(\frac{b}{y}\right)\right)\right)$
			$= \text{int}\left(\frac{ab}{y} - \frac{ab}{y} + a \times \text{int}\left(\frac{b}{y}\right)\right) + \text{int}\left(\frac{ab}{y} - a \times \text{int}\left(\frac{b}{y}\right)\right)$
			$= \text{int}\left(a \times \text{int}\left(\frac{b}{y}\right)\right) + \text{int}\left(\frac{ab}{y} - a \times \text{int}\left(\frac{b}{y}\right)\right)$
R4	R1	$\Rightarrow$	$a \times \text{int}\left(\frac{b}{y}\right) = \text{int}\left(a \times \text{int}\left(\frac{b}{y}\right)\right)$
R5	R3 & R4	$\Rightarrow$	$x = \text{int}\left(a \times \text{int}\left(\frac{b}{y}\right)\right) + \text{int}\left(\frac{ab}{y} - \text{int}\left(a \times \text{int}\left(\frac{b}{y}\right)\right)\right)$
R6	R5 & 28	$\Rightarrow$	$x = \text{int}\left(a \times \text{int}\left(\frac{b}{y}\right)\right) + \text{int}\left(\frac{ab}{y}\right) - \text{int}\left(a \times \text{int}\left(\frac{b}{y}\right)\right) = \text{int}\left(\frac{ab}{y}\right)$
R7	R6 & 48	$\Rightarrow$	$x = (ab) \text{ div } y$
R8	R7 & R2	$\Rightarrow$	$(ab - a \times (b \bmod y)) \text{ div } y + (a \times (b \bmod y)) \text{ div } y = ab \text{ div } y$

Theorem 58 *If a and b are integers and y is a non-zero integer then*

$$ab \text{ div } y = a \times (b \text{ div } y) + (a \times (b \bmod y)) \text{ div } y$$

Proof

R1	Let		a and b be integers and y be a non-zero integer
R2	49	$\Rightarrow$	$b = b \bmod y + y \times (b \operatorname{div} y)$
		$\Rightarrow$	$\frac{ab}{y} = \frac{a}{y} \times (b \bmod y) + a \times (b \operatorname{div} y)$
		$\Rightarrow$	$a \times (b \operatorname{div} y) = \frac{ab}{y} - \frac{a}{y} \times (b \bmod y)$
R3	R1 & 48	$\Rightarrow$	$a \times (b \operatorname{div} y)$ is an integer
R4	R2 & R3	$\Rightarrow$	$\frac{ab}{y} - \frac{a}{y} \times (b \bmod y)$ is an integer
		$\Rightarrow$	$\frac{ab}{y} - \frac{a}{y} \times (b \bmod y) = \operatorname{int}\left(\frac{ab}{y} - \frac{a}{y} \times (b \bmod y)\right)$
R5	R2 & R4	$\Rightarrow$	$a \times (b \operatorname{div} y) = \operatorname{int}\left(\frac{ab}{y} - \frac{a}{y} \times (b \bmod y)\right)$
R6	48 & R5	$\Rightarrow$	$a \times (b \operatorname{div} y) = (ab - a \times (b \bmod y)) \operatorname{div} y$
R7	R6	$\Rightarrow$	$a \times (b \operatorname{div} y) + (a \times (b \bmod y)) \operatorname{div} y = (ab - a \times (b \bmod y)) \operatorname{div} y + (a \times (b \bmod y)) \operatorname{div} y$
R8	R7 & 57	$\Rightarrow$	$a \times (b \operatorname{div} y) + (a \times (b \bmod y)) \operatorname{div} y = ab \operatorname{div} y$

4.2.4 log

Theorem 59 *If a, b and c are any three positive real numbers then*

$$\log_a b \times \log_b c = \log_a c$$

Proof

R1	Let		$c = a^x = b^y$
R2	R1	$\Rightarrow$	$x = y \log_a b$
R3	R1	$\Rightarrow$	$x = \log_a c$
R4	R1	$\Rightarrow$	$y = \log_b c$
R5	R2 & R4	$\Rightarrow$	$x = \log_b c \times \log_a b$
R6	R3 & R5	$\Rightarrow$	$\log_a c = \log_a b \times \log_b c$

4.2.5 abs

Definition 60 (abs) *If x is a real number then*

$$\mathrm{abs}\,(x) = \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \text{if} \quad x < 0 \end{cases}$$

4.3 *MIPS* objects

4.3.1 Pitch system and pitch: the primary *MIPS* concepts

Definition 61 (Pitch system) *An object ψ is a well-formed* pitch system *if and only if it is an ordered quadruple*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

such that the following conditions are satisfied:

1. μ_{c} *is a natural number called the* chromatic modulus*;*
2. μ_{m} *is a natural number called the* morphetic modulus*;*
3. $\mu_{\mathrm{c}} \geq \mu_{\mathrm{m}}$*;*
4. f_0 *is a positive real number called the* standard frequency*;*
5. $p_{\mathrm{c},0}$ *is an integer called the* standard chromatic pitch*.*

Definition 62 (Pitch) *An object p is a well-formed* pitch *in a pitch system if and only if it is an ordered pair*

$$p = [p_{\mathrm{c}}, p_{\mathrm{m}}]$$

that satisfies the following conditions:

1. p_{c} *is an integer called the* chromatic pitch*;*
2. p_{m} *is an integer called the* morphetic pitch*.*

4.3.2 Derived *MIPS* objects

Deriving objects from a *MIPS* pitch

Definition 63 (Chromatic pitch of a pitch) *If $p = [p_{\mathrm{c}}, p_{\mathrm{m}}]$ is a pitch in a well-formed pitch system then the following function returns the* chromatic pitch *of p:*

$$\mathrm{p_c}\,(p) = p_{\mathrm{c}}$$

Definition 64 (Morphetic pitch of a pitch) *If $p = [p_{\mathrm{c}}, p_{\mathrm{m}}]$ is a pitch in a well-formed pitch system then the following function returns the* morphetic pitch *of p:*

$$\mathrm{p_m}\,(p) = p_{\mathrm{m}}$$

Theorem 65 *If ψ is a pitch system and p is a pitch in ψ then*

$$p = [\mathrm{p_c}\,(p)\,, \mathrm{p_m}\,(p)]$$

Proof

R1	Let		$p = [p_c, p_m]$
R2	R1 & 63	$\Rightarrow$	$\mathrm{p_c}(p) = p_c$
R3	R1 & 64	$\Rightarrow$	$\mathrm{p_m}(p) = p_m$
R4	R1, R2 & R3	$\Rightarrow$	$p = [\mathrm{p_c}(p), \mathrm{p_m}(p)]$

Definition 66 (Frequency of a pitch) *If* p *is a pitch in the pitch system*

$$\psi = [\mu_c, \mu_m, f_0, p_{c,0}]$$

then the function

$$\mathrm{f}(p) = f_0 \times 2^{(\mathrm{p_c}(p) - p_{c,0})/\mu_c}$$

returns the frequency *of* p*.*

Theorem 67 *If* f *is the frequency of a pitch* p *in a pitch system* ψ *then* f *can only take any value such that*

$$f \in \mathbb{R}^+$$

where $\mathbb{R}^+$ *is the universal set of real numbers greater than zero.*

Proof

R1	Let		p be any pitch in $\psi = [\mu_c, \mu_m, f_0, p_{c,0}]$
R2	Let		$f = \mathrm{f}(p)$
R3	66 & R2	$\Rightarrow$	$f = f_0 \times 2^{(\mathrm{p_c}(p) - p_{c,0})/\mu_c}$
R4	61	$\Rightarrow$	f_0 can only take any positive real value.
R5			2^x can only take any positive real value when x is real.
R6	R3, R4 & R5	$\Rightarrow$	f can only take any value such that $f \in \mathbb{R}^+$
			where $\mathbb{R}^+$ is the universal set of real numbers greater than zero.

Definition 68 (Chromatic octave of a pitch) *If* p *is a pitch in the pitch system*

$$\psi = [\mu_c, \mu_m, f_0, p_{c,0}]$$

then the following function returns the chromatic octave *of* p*:*

$$\mathrm{o_c}(p) = \mathrm{p_c}(p) \text{ div } \mu_c$$

Definition 69 (Morphetic octave of a pitch) *If p is a pitch in the pitch system*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

then the following function returns the morphetic octave *of p:*

$$\mathrm{o}_{\mathrm{m}}(p) = \mathrm{p}_{\mathrm{m}}(p) \text{ div } \mu_{\mathrm{m}}$$

Theorem 70 ($\mathrm{o}_{\mathrm{m}}(p) \in \mathbb{Z}$) *If p is a pitch in a pitch system ψ then*

$$\mathrm{o}_{\mathrm{m}}(p) \in \mathbb{Z}$$

where $\mathbb{Z}$ is the universal set of integers.

Proof

R1	69	$\Rightarrow$	$\mathrm{o}_{\mathrm{m}}(p) = \mathrm{p}_{\mathrm{m}}(p) \text{ div } \mu_{\mathrm{m}}$
R2	R1 & 48	$\Rightarrow$	$\mathrm{o}_{\mathrm{m}}(p) = \mathrm{int}(\mathrm{p}_{\mathrm{m}}(p)/\mu_{\mathrm{m}})$
R3	R2 & 27	$\Rightarrow$	$\mathrm{o}_{\mathrm{m}}(p) \in \mathbb{Z}$ where $\mathbb{Z}$ is the universal set of integers

Definition 71 (Chroma of a pitch) *If p is a pitch in a pitch system*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

then the following function returns the chroma *of p:*

$$\mathrm{c}(p) = \mathrm{p}_{\mathrm{c}}(p) \bmod \mu_{\mathrm{c}}$$

Theorem 72 *If c is the chroma of a pitch p in a pitch system*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

then c can only take any value such that

$$(0 \leq c < \mu_{\mathrm{c}}) \wedge (c \in \mathbb{Z})$$

where $\mathbb{Z}$ is the universal set of integers.

Proof

R1	Let		$c = \mathrm{c}(p)$
R2	71	$\Rightarrow$	$\mathrm{c}(p) = \mathrm{p_c}(p) \bmod \mu_\mathrm{c}$
R3	R1 & R2	$\Rightarrow$	$c = \mathrm{p_c}(p) \bmod \mu_\mathrm{c}$
R4	61	$\Rightarrow$	μ_c can only take any positive integer value.
R5	R4 & 41	$\Rightarrow$	$\mu_\mathrm{c} > \mathrm{p_c}(p) \bmod \mu_\mathrm{c} \geq 0$
R6	R3 & R5	$\Rightarrow$	$\mu_\mathrm{c} > c \geq 0$
R7	63 & 62	$\Rightarrow$	$\mathrm{p_c}(p)$ can only take any integer value.
R8	R3 & 33	$\Rightarrow$	$c = \mathrm{p_c}(p) - \mu_\mathrm{c} \times \mathrm{int}\left(\frac{\mathrm{p_c}(p)}{\mu_\mathrm{c}}\right)$
R9	R8, R7, R4 & 27	$\Rightarrow$	c is an integer
R10	R9 & R6	$\Rightarrow$	$(0 \leq c < \mu_\mathrm{c}) \wedge (c \in \mathbb{Z})$ where $\mathbb{Z}$ is the universal set of integers.
R11	R7	$\Rightarrow$	$\mathrm{p_c}(p)$ can take any integer value such that $\mu_\mathrm{c} > \mathrm{p_c}(p) \geq 0$.
R12	45 & R3	$\Rightarrow$	$c = \mathrm{p_c}(p)$ for each value of $\mathrm{p_c}(p)$ such that $\mu_\mathrm{c} > \mathrm{p_c}(p) \geq 0$.
R13	R11 & R12	$\Rightarrow$	c can take any integer value such that $\mu_\mathrm{c} > c \geq 0$.
R14	R13 & R10	$\Rightarrow$	c can only take any value such that $(0 \leq c < \mu_\mathrm{c}) \wedge (c \in \mathbb{Z})$.

Theorem 73 *If*

$$\psi = [\mu_\mathrm{c}, \mu_\mathrm{m}, f_0, p_{\mathrm{c},0}]$$

is a pitch system and c *is a chroma in* ψ *then*

$$c \bmod \mu_\mathrm{c} = c$$

Proof

R1	72	$\Rightarrow$	$(0 \leq c < \mu_\mathrm{c}) \wedge (c \in \mathbb{Z})$
R2	R1 & 44	$\Rightarrow$	$c \bmod \mu_\mathrm{c} = c$

Theorem 74 *If*

$$\psi = [\mu_\mathrm{c}, \mu_\mathrm{m}, f_0, p_{\mathrm{c},0}]$$

is a pitch system and c *is a chroma in* ψ *then*

$$c \operatorname{div} \mu_\mathrm{c} = 0$$

Proof

R1	72	$\Rightarrow$	$(0 \leq c < \mu_{\mathrm{c}}) \wedge (c \in \mathbb{Z})$
R2	48	$\Rightarrow$	$c \text{ div } \mu_{\mathrm{c}} = \operatorname{int}\left(\frac{c}{\mu_{\mathrm{c}}}\right)$
R3	R1 & R2	$\Rightarrow$	$c \text{ div } \mu_{\mathrm{c}} = 0$

Theorem 75 *If $\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$ is a pitch system and p is a pitch in ψ then*

$$\mathrm{p_c}(p) = \mathrm{c}(p) + \mathrm{o_c}(p) \times \mu_{\mathrm{c}}$$

Proof

R1	68	$\Rightarrow$	$\mathrm{o_c}(p) = \mathrm{p_c}(p) \text{ div } \mu_{\mathrm{c}}$
R2	71	$\Rightarrow$	$\mathrm{c}(p) = \mathrm{p_c}(p) \bmod \mu_{\mathrm{c}}$
R3	49, 63 & 61	$\Rightarrow$	$\mathrm{p_c}(p) = \mathrm{p_c}(p) \bmod \mu_{\mathrm{c}} + \mu_{\mathrm{c}} \times (\mathrm{p_c}(p) \text{ div } \mu_{\mathrm{c}})$
R4	R1, R2 & R3	$\Rightarrow$	$\mathrm{p_c}(p) = \mathrm{c}(p) + \mathrm{o_c}(p) \times \mu_{\mathrm{c}}$

Definition 76 (Morph of a pitch) *If p is a pitch in the pitch system*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

then the following function returns the morph *of p:*

$$\mathrm{m}(p) = \mathrm{p_m}(p) \bmod \mu_{\mathrm{m}}$$

Theorem 77 *If m is the morph of a pitch p in a pitch system*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

then m can only take any value such that

$$(0 \leq m < \mu_{\mathrm{m}}) \wedge (m \in \mathbb{Z})$$

where $\mathbb{Z}$ is the universal set of integers.

Proof

R1	Let		$m = \mathrm{m}(p)$
R2	76	$\Rightarrow$	$\mathrm{m}(p) = \mathrm{p_m}(p) \bmod \mu_\mathrm{m}$
R3	R1 & R2	$\Rightarrow$	$m = \mathrm{p_m}(p) \bmod \mu_\mathrm{m}$
R4	61	$\Rightarrow$	μ_m can only take any positive integer value.
R5	R4 & 41	$\Rightarrow$	$\mu_\mathrm{m} > \mathrm{p_m}(p) \bmod \mu_\mathrm{m} \geq 0$
R6	R3 & R5	$\Rightarrow$	$\mu_\mathrm{m} > m \geq 0$
R7	64 & 62	$\Rightarrow$	$\mathrm{p_m}(p)$ can only take any integer value.
R8	R3 & 33	$\Rightarrow$	$m = \mathrm{p_m}(p) - \mu_\mathrm{m} \times \mathrm{int}\left(\frac{\mathrm{p_m}(p)}{\mu_\mathrm{m}}\right)$
R9	R8, R7, R4 & 27	$\Rightarrow$	m is an integer
R10	R9 & R6	$\Rightarrow$	$(0 \leq m < \mu_\mathrm{m}) \wedge (m \in \mathbb{Z})$ where $\mathbb{Z}$ is the universal set of integers.
R11	R7	$\Rightarrow$	$\mathrm{p_m}(p)$ can take any integer value such that $\mu_\mathrm{m} > \mathrm{p_m}(p) \geq 0$.
R12	45 & R3	$\Rightarrow$	$m = \mathrm{p_m}(p)$ for each value of $\mathrm{p_m}(p)$ such that $\mu_\mathrm{m} > \mathrm{p_m}(p) \geq 0$.
R13	R11 & R12	$\Rightarrow$	m can take any integer value such that $\mu_\mathrm{m} > m \geq 0$.
R14	R13 & R10	$\Rightarrow$	m can only take any value such that $(0 \leq m < \mu_\mathrm{m}) \wedge (m \in \mathbb{Z})$.

Theorem 78 *If*

$$\psi = [\mu_\mathrm{c}, \mu_\mathrm{m}, f_0, p_{\mathrm{c},0}]$$

is a pitch system and m *is a morph in* ψ *then*

$$m \bmod \mu_\mathrm{m} = m$$

Proof

R1	77	$\Rightarrow$	$(0 \leq m < \mu_\mathrm{m}) \wedge (m \in \mathbb{Z})$
R2	R1 & 44	$\Rightarrow$	$m \bmod \mu_\mathrm{m} = m$

Theorem 79 *If*

$$\psi = [\mu_\mathrm{c}, \mu_\mathrm{m}, f_0, p_{\mathrm{c},0}]$$

is a pitch system and m *is a morph in* ψ *then*

$$m \operatorname{div} \mu_\mathrm{m} = 0$$

Proof

R1	77	$\Rightarrow$	$(0 \leq m < \mu_{\mathrm{m}}) \wedge (m \in \mathbb{Z})$
R2	48	$\Rightarrow$	$m \operatorname{div} \mu_{\mathrm{m}} = \operatorname{int}\left(\frac{m}{\mu_{\mathrm{m}}}\right)$
R3	R1 & R2	$\Rightarrow$	$m \operatorname{div} \mu_{\mathrm{m}} = 0$

Definition 80 (Chromamorph of a pitch) *If p is a pitch in a well-formed pitch system, then the following function returns the* chromamorph *of p:*

$$\mathrm{q}(p) = [\mathrm{c}(p), \mathrm{m}(p)]$$

Definition 81 (Octave difference of a pitch) *If p is a pitch in a well-formed pitch system, then the following function returns the* octave difference *of p:*

$$\mathrm{d_o}(p) = \mathrm{o_c}(p) - \mathrm{o_m}(p)$$

Definition 82 (Chromatic genus of a pitch) *If p is a pitch in a well-formed pitch system*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

then the following function returns the chromatic genus *of p:*

$$\mathrm{g_c}(p) = \mathrm{p_c}(p) - \mu_{\mathrm{c}} \times \mathrm{o_m}(p)$$

Theorem 83 *If p is any pitch in a pitch system ψ then $\mathrm{g_c}(p)$ can only take any integer value.*

Proof

R1	Let		p be any pitch in ψ.
R2	82	$\Rightarrow$	$\mathrm{g_c}(p) = \mathrm{p_c}(p) - \mu_{\mathrm{c}} \times \mathrm{o_m}(p)$
R3	62 & 63	$\Rightarrow$	$\mathrm{p_c}(p)$ can only take any integer value.
R4	61	$\Rightarrow$	μ_{c} can only take any positive integer value.
R5	70	$\Rightarrow$	$\mathrm{o_m}(p)$ is an integer.
R6	63, 69 & 61	$\Rightarrow$	μ_{c}, $\mathrm{p_c}(p)$ and $\mathrm{o_m}(p)$ are mutually independent values.
R7	R2 to R6	$\Rightarrow$	$\mathrm{g_c}(p)$ can only take any integer value.

Definition 84 (Genus of a pitch) *If p is a pitch in a well-formed pitch system then the following function returns the* genus *of p:*

$$\mathrm{g}(p) = [\mathrm{g_c}(p), \mathrm{m}(p)]$$

Theorem 85 *If p_1 and p_2 are two pitches in a pitch system ψ then*

$$(\mathrm{d_o}(p_1) = \mathrm{d_o}(p_2)) \wedge (\mathrm{c}(p_1) = \mathrm{c}(p_2)) \wedge (\mathrm{m}(p_1) = \mathrm{m}(p_2)) \Rightarrow (\mathrm{g}(p_1) = \mathrm{g}(p_2))$$

Proof

R1	81	$\Rightarrow$	$(\mathrm{d_o}(p_1) = \mathrm{d_o}(p_2)) \Rightarrow (\mathrm{o_c}(p_1) - \mathrm{o_m}(p_1) = \mathrm{o_c}(p_2) - \mathrm{o_m}(p_2))$
R2	R1 & 75	$\Rightarrow$	$\left((\mathrm{d_o}(p_1) = \mathrm{d_o}(p_2)) \Rightarrow \left(\frac{\mathrm{p_c}(p_1)-\mathrm{c}(p_1)}{\mu_\mathrm{c}} - \mathrm{o_m}(p_1) = \frac{\mathrm{p_c}(p_2)-\mathrm{c}(p_2)}{\mu_\mathrm{c}} - \mathrm{o_m}(p_2)\right)\right)$
		$\Rightarrow$	$((\mathrm{d_o}(p_1) = \mathrm{d_o}(p_2)) \Rightarrow (\mathrm{p_c}(p_1) - \mathrm{c}(p_1) - \mu_\mathrm{c} \times \mathrm{o_m}(p_1) = \mathrm{p_c}(p_2) - \mathrm{c}(p_2) - \mu_\mathrm{c} \times \mathrm{o_m}(p_2)))$
		$\Rightarrow$	$((\mathrm{d_o}(p_1) = \mathrm{d_o}(p_2) \wedge \mathrm{c}(p_1) = \mathrm{c}(p_2)) \Rightarrow (\mathrm{p_c}(p_1) - \mu_\mathrm{c} \times \mathrm{o_m}(p_1) = \mathrm{p_c}(p_2) - \mu_\mathrm{c} \times \mathrm{o_m}(p_2)))$
R3	R2 & 82	$\Rightarrow$	$((\mathrm{d_o}(p_1) = \mathrm{d_o}(p_2) \wedge \mathrm{c}(p_1) = \mathrm{c}(p_2)) \Rightarrow (\mathrm{g_c}(p_1) = \mathrm{g_c}(p_2)))$
R4	R3 & 84	$\Rightarrow$	$((\mathrm{d_o}(p_1) = \mathrm{d_o}(p_2) \wedge \mathrm{c}(p_1) = \mathrm{c}(p_2) \wedge \mathrm{m}(p_1) = \mathrm{m}(p_2)) \Rightarrow (\mathrm{g}(p_1) = \mathrm{g}(p_2)))$

Theorem 86 *If p_1 and p_2 are two pitches in a pitch system ψ then*

$$\mathrm{g}(p_1) = \mathrm{g}(p_2) \Rightarrow \mathrm{d_o}(p_1) = \mathrm{d_o}(p_2) \wedge \mathrm{c}(p_1) = \mathrm{c}(p_2) \wedge \mathrm{m}(p_1) = \mathrm{m}(p_2)$$

Proof

R1	84	$\Rightarrow$	$(\mathrm{g}(p_1) = \mathrm{g}(p_2) \Rightarrow [\mathrm{g_c}(p_1), \mathrm{m}(p_1)] = [\mathrm{g_c}(p_2), \mathrm{m}(p_2)])$
R2	R1	$\Rightarrow$	$(\mathrm{g}(p_1) = \mathrm{g}(p_2) \Rightarrow \mathrm{m}(p_1) = \mathrm{m}(p_2))$
R3	R1	$\Rightarrow$	$(\mathrm{g}(p_1) = \mathrm{g}(p_2) \Rightarrow \mathrm{g_c}(p_1) = \mathrm{g_c}(p_2))$
R4	R3 & 82	$\Rightarrow$	$(\mathrm{g}(p_1) = \mathrm{g}(p_2) \Rightarrow \mathrm{p_c}(p_1) - \mu_\mathrm{c} \times \mathrm{o_m}(p_1) = \mathrm{p_c}(p_2) - \mu_\mathrm{c} \times \mathrm{o_m}(p_2))$
R5	R4 & 47	$\Rightarrow$	$(\mathrm{g}(p_1) = \mathrm{g}(p_2) \Rightarrow \mathrm{p_c}(p_1) \bmod \mu_\mathrm{c} = \mathrm{p_c}(p_2) \bmod \mu_\mathrm{c})$
R6	R5 & 71	$\Rightarrow$	$(\mathrm{g}(p_1) = \mathrm{g}(p_2) \Rightarrow \mathrm{c}(p_1) = \mathrm{c}(p_2))$
R7	R4 & R6	$\Rightarrow$	$(\mathrm{g}(p_1) = \mathrm{g}(p_2) \Rightarrow \mathrm{p_c}(p_1) - \mathrm{c}(p_1) - \mu_\mathrm{c} \times \mathrm{o_m}(p_1) = \mathrm{p_c}(p_2) - \mathrm{c}(p_2) - \mu_\mathrm{c} \times \mathrm{o_m}(p_2))$
		$\Rightarrow$	$\left(\mathrm{g}(p_1) = \mathrm{g}(p_2) \Rightarrow \frac{\mathrm{p_c}(p_1)-\mathrm{c}(p_1)}{\mu_\mathrm{c}} - \mathrm{o_m}(p_1) = \frac{\mathrm{p_c}(p_2)-\mathrm{c}(p_2)}{\mu_\mathrm{c}} - \mathrm{o_m}(p_2)\right)$
R8	R7 & 75	$\Rightarrow$	$(\mathrm{g}(p_1) = \mathrm{g}(p_2) \Rightarrow \mathrm{o_c}(p_1) - \mathrm{o_m}(p_1) = \mathrm{o_c}(p_2) - \mathrm{o_m}(p_2))$
R9	R8 & 81	$\Rightarrow$	$(\mathrm{g}(p_1) = \mathrm{g}(p_2) \Rightarrow \mathrm{d_o}(p_1) = \mathrm{d_o}(p_2))$
R10	R2, R6 & R9	$\Rightarrow$	$(\mathrm{g}(p_1) = \mathrm{g}(p_2) \Rightarrow \mathrm{d_o}(p_1) = \mathrm{d_o}(p_2) \wedge \mathrm{c}(p_1) = \mathrm{c}(p_2) \wedge \mathrm{m}(p_1) = \mathrm{m}(p_2))$

Theorem 87 *If* p_1 *and* p_2 *are two pitches in a pitch system* ψ *then*

$$\mathrm{g}(p_1) = \mathrm{g}(p_2) \iff \mathrm{d_o}(p_1) = \mathrm{d_o}(p_2) \wedge \mathrm{c}(p_1) = \mathrm{c}(p_2) \wedge \mathrm{m}(p_1) = \mathrm{m}(p_2)$$

Proof

R1	85	$\Rightarrow$	$(\mathrm{d_o}(p_1) = \mathrm{d_o}(p_2) \wedge \mathrm{c}(p_1) = \mathrm{c}(p_2) \wedge \mathrm{m}(p_1) = \mathrm{m}(p_2) \Rightarrow \mathrm{g}(p_1) = \mathrm{g}(p_2))$
R2	86	$\Rightarrow$	$(\mathrm{g}(p_1) = \mathrm{g}(p_2) \Rightarrow \mathrm{d_o}(p_1) = \mathrm{d_o}(p_2) \wedge \mathrm{c}(p_1) = \mathrm{c}(p_2) \wedge \mathrm{m}(p_1) = \mathrm{m}(p_2))$
R3	R1 & R2	$\Rightarrow$	$(\mathrm{g}(p_1) = \mathrm{g}(p_2) \iff \mathrm{d_o}(p_1) = \mathrm{d_o}(p_2) \wedge \mathrm{c}(p_1) = \mathrm{c}(p_2) \wedge \mathrm{m}(p_1) = \mathrm{m}(p_2))$

Deriving *MIPS* objects from a chromatic pitch

Definition 88 (Definition of $\mathrm{f}(p_\mathrm{c})$) *If* p_c *is the chromatic pitch of a pitch* p *in a pitch system* ψ *then the function* $\mathrm{f}(p_\mathrm{c})$ *must return the frequency of* p*. In other words, by definition, it must be true that*

$$(p_\mathrm{c} = \mathrm{p_c}(p)) \Rightarrow (\mathrm{f}(p_\mathrm{c}) = \mathrm{f}(p))$$

Theorem 89 (Formula for $\mathrm{f}(p_\mathrm{c})$) *If* p_c *is the chromatic pitch of a pitch in*

$$\psi = [\mu_\mathrm{c}, \mu_\mathrm{m}, f_0, p_{\mathrm{c},0}]$$

then

$$\mathrm{f}(p_\mathrm{c}) = f_0 \times 2^{(p_\mathrm{c} - p_{\mathrm{c},0})/\mu_\mathrm{c}}$$

Proof

R1	Let		$p_\mathrm{c} = \mathrm{p_c}(p)$
R2	66	$\Rightarrow$	$\mathrm{f}(p) = f_0 \times 2^{(\mathrm{p_c}(p) - p_{\mathrm{c},0})/\mu_\mathrm{c}}$
R3	R1 & R2	$\Rightarrow$	$\mathrm{f}(p) = f_0 \times 2^{(p_\mathrm{c} - p_{\mathrm{c},0})/\mu_\mathrm{c}}$
R4	R1, R3 & 88	$\Rightarrow$	$\mathrm{f}(p_\mathrm{c}) = f_0 \times 2^{(p_\mathrm{c} - p_{\mathrm{c},0})/\mu_\mathrm{c}}$

Definition 90 (Definition of $\mathrm{o_c}(p_\mathrm{c})$) *If* p_c *is the chromatic pitch of a pitch* p *in a pitch system* ψ *then the function* $\mathrm{o_c}(p_\mathrm{c})$ *must return the chromatic octave of* p*. In other words, by definition, it must be true that*

$$(p_\mathrm{c} = \mathrm{p_c}(p)) \Rightarrow (\mathrm{o_c}(p_\mathrm{c}) = \mathrm{o_c}(p))$$

Theorem 91 (Formula for $\mathrm{o_c}(p_\mathrm{c})$) *If* p_c *is the chromatic pitch of a pitch in*

$$\psi = [\mu_\mathrm{c}, \mu_\mathrm{m}, f_0, p_{\mathrm{c},0}]$$

then

$$\mathrm{o_c}(p_\mathrm{c}) = p_\mathrm{c} \ \mathrm{div}\ \mu_\mathrm{c}$$

Proof

R1	Let		$p_{\mathrm{c}} = \mathrm{p_c}(p)$
R2	68	$\Rightarrow$	$\mathrm{o_c}(p) = \mathrm{p_c}(p) \text{ div } \mu_{\mathrm{c}}$
R3	R1 & R2	$\Rightarrow$	$\mathrm{o_c}(p) = p_{\mathrm{c}} \text{ div } \mu_{\mathrm{c}}$
R4	R1, R3 & 90	$\Rightarrow$	$\mathrm{o_c}(p_{\mathrm{c}}) = p_{\mathrm{c}} \text{ div } \mu_{\mathrm{c}}$

Definition 92 (Definition of $\mathrm{c}(p_{\mathrm{c}})$) *If p_{c} is the chromatic pitch of a pitch p in a pitch system ψ then the function $\mathrm{c}(p_{\mathrm{c}})$ must return the chroma of p. In other words, by definition, it must be true that*

$$(p_{\mathrm{c}} = \mathrm{p_c}(p)) \Rightarrow (\mathrm{c}(p_{\mathrm{c}}) = \mathrm{c}(p))$$

Theorem 93 (Formula for $\mathrm{c}(p_{\mathrm{c}})$) *If p_{c} is the chromatic pitch of a pitch in the pitch system*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

then:

$$\mathrm{c}(p_{\mathrm{c}}) = p_{\mathrm{c}} \text{ mod } \mu_{\mathrm{c}}$$

Proof

R1	Let		$p_{\mathrm{c}} = \mathrm{p_c}(p)$
R2	71	$\Rightarrow$	$\mathrm{c}(p) = \mathrm{p_c}(p) \text{ mod } \mu_{\mathrm{c}}$
R3	R1 & R2	$\Rightarrow$	$\mathrm{c}(p) = p_{\mathrm{c}} \text{ mod } \mu_{\mathrm{c}}$
R4	R1, R3 & 92	$\Rightarrow$	$\mathrm{c}(p_{\mathrm{c}}) = p_{\mathrm{c}} \text{ mod } \mu_{\mathrm{c}}$

Deriving *MIPS* objects from a morphetic pitch

Definition 94 (Definition of $\mathrm{o_m}(p_{\mathrm{m}})$) *If p_{m} is the morphetic pitch of a pitch p in a pitch system ψ then the function $\mathrm{o_m}(p_{\mathrm{m}})$ must return the morphetic octave of p. In other words, by definition, it must be true that*

$$(p_{\mathrm{m}} = \mathrm{p_m}(p)) \Rightarrow (\mathrm{o_m}(p_{\mathrm{m}}) = \mathrm{o_m}(p))$$

Theorem 95 (Formula for $\mathrm{o_m}(p_{\mathrm{m}})$) *If p_{m} is the morphetic pitch of a pitch in*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

then

$$\mathrm{o_m}(p_{\mathrm{m}}) = p_{\mathrm{m}} \text{ div } \mu_{\mathrm{m}}$$

Proof

R1	Let		$p_{\mathrm{m}} = \mathrm{p_m}(p)$
R2	69	$\Rightarrow$	$\mathrm{o_m}(p) = \mathrm{p_m}(p) \text{ div } \mu_{\mathrm{m}}$
R3	R1 & R2	$\Rightarrow$	$\mathrm{o_m}(p) = p_{\mathrm{m}} \text{ div } \mu_{\mathrm{m}}$
R4	R1, R3 & 94	$\Rightarrow$	$\mathrm{o_m}(p_{\mathrm{m}}) = p_{\mathrm{m}} \text{ div } \mu_{\mathrm{m}}$

Definition 96 (Definition of $\mathrm{m}(p_{\mathrm{m}})$) *If p_{m} is the morphetic pitch of a pitch p in a pitch system ψ then the function $\mathrm{m}(p_{\mathrm{m}})$ must return the morph of p. In other words, by definition, it must be true that*

$$(p_{\mathrm{m}} = \mathrm{p_m}(p)) \Rightarrow (\mathrm{m}(p_{\mathrm{m}}) = \mathrm{m}(p))$$

Theorem 97 (Formula for $\mathrm{m}(p_{\mathrm{m}})$) *If p_{m} is the morphetic pitch of a pitch in the pitch system*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

then:

$$\mathrm{m}(p_{\mathrm{m}}) = p_{\mathrm{m}} \bmod \mu_{\mathrm{m}}$$

Proof

R1	Let		$p_{\mathrm{m}} = \mathrm{p_m}(p)$
R2	76	$\Rightarrow$	$\mathrm{m}(p) = \mathrm{p_m}(p) \bmod \mu_{\mathrm{m}}$
R3	R1 & R2	$\Rightarrow$	$\mathrm{m}(p) = p_{\mathrm{m}} \bmod \mu_{\mathrm{m}}$
R4	R1, R3 & 96	$\Rightarrow$	$\mathrm{m}(p_{\mathrm{m}}) = p_{\mathrm{m}} \bmod \mu_{\mathrm{m}}$

Deriving *MIPS* objects from a frequency

Definition 98 (Definition of $\mathrm{p_c}(f)$) *If f is the frequency of a pitch p in a pitch system ψ then the function $\mathrm{p_c}(f)$ must return the chromatic pitch of p. In other words, by definition, it must be true that*

$$(f = \mathrm{f}(p)) \Rightarrow (\mathrm{p_c}(f) = \mathrm{p_c}(p))$$

Theorem 99 (Formula for $\mathrm{p_c}(f)$) *If f is the frequency of a pitch in the pitch system*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

then

$$\mathrm{p_c}(f) = \mu_{\mathrm{c}} \times \frac{\ln(f/f_0)}{\ln 2} + p_{\mathrm{c},0}$$

Proof

R1	Let		$f = \mathrm{f}(p)$
R2	66	$\Rightarrow$	$\mathrm{f}(p) = f_0 \times 2^{(\mathrm{p_c}(p) - p_{\mathrm{c},0})/\mu_{\mathrm{c}}}$
			$\Rightarrow \log_2(\mathrm{f}(p)) = \log_2 f_0 + \frac{\mathrm{p_c}(p) - p_{\mathrm{c},0}}{\mu_{\mathrm{c}}}$
			$\Rightarrow \mathrm{p_c}(p) = \mu_{\mathrm{c}} \times \log_2(\mathrm{f}(p)/f_0) + p_{\mathrm{c},0}$
R3	R2 & 59	$\Rightarrow$	$\mathrm{p_c}(p) = \mu_{\mathrm{c}} \times \frac{\ln(\mathrm{f}(p)/f_0)}{\ln 2} + p_{\mathrm{c},0}$
R4	R3 & R1	$\Rightarrow$	$\mathrm{p_c}(p) = \mu_{\mathrm{c}} \times \frac{\ln(f/f_0)}{\ln 2} + p_{\mathrm{c},0}$
R5	R4, R1 & 98	$\Rightarrow$	$\mathrm{p_c}(f) = \mu_{\mathrm{c}} \times \frac{\ln(f/f_0)}{\ln 2} + p_{\mathrm{c},0}$

Theorem 100 (Second formula for $\mathrm{p_c}(f)$) *If f is the frequency of a pitch in the pitch system*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

then

$$\mathrm{p_c}(f) = \mu_{\mathrm{c}} \times \log_2(f/f_0) + p_{\mathrm{c},0}$$

Proof

R1	Let		$f = \mathrm{f}(p)$
R2	66	$\Rightarrow$	$\mathrm{f}(p) = f_0 \times 2^{(\mathrm{p_c}(p) - p_{\mathrm{c},0})/\mu_{\mathrm{c}}}$
			$\Rightarrow \log_2(\mathrm{f}(p)) = \log_2 f_0 + \frac{\mathrm{p_c}(p) - p_{\mathrm{c},0}}{\mu_{\mathrm{c}}}$
			$\Rightarrow \mathrm{p_c}(p) = \mu_{\mathrm{c}} \times \log_2(\mathrm{f}(p)/f_0) + p_{\mathrm{c},0}$
R3	R2, R1 & 98	$\Rightarrow$	$\mathrm{p_c}(f) = \mu_{\mathrm{c}} \times \log_2(f/f_0) + p_{\mathrm{c},0}$

Definition 101 (Definition of $\mathrm{o_c}(f)$) *If f is the frequency of a pitch p in a pitch system ψ then the function $\mathrm{o_c}(f)$ must return the chromatic octave of p. In other words, by definition, it must be true that*

$$(f = \mathrm{f}(p)) \Rightarrow (\mathrm{o_c}(f) = \mathrm{o_c}(p))$$

Theorem 102 (Formula for $\mathrm{o_c}(f)$) *If f is the frequency of a pitch in the pitch system*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

then

$$\mathrm{o_c}(f) = \mathrm{p_c}(f) \;\mathrm{div}\; \mu_{\mathrm{c}}$$

Proof

R1	Let		$f = \mathrm{f}(p)$
R2	68	$\Rightarrow$	$\mathrm{o_c}(p) = \mathrm{p_c}(p) \operatorname{div} \mu_\mathrm{c}$
R3	R1 & 98	$\Rightarrow$	$\mathrm{o_c}(p) = \mathrm{p_c}(f) \operatorname{div} \mu_\mathrm{c}$
R4	R1, R3 & 101	$\Rightarrow$	$\mathrm{o_c}(f) = \mathrm{p_c}(f) \operatorname{div} \mu_\mathrm{c}$

Definition 103 (Definition of $\mathrm{c}(f)$) *If f is the frequency of a pitch p in a pitch system ψ then the function $\mathrm{c}(f)$ must return the chroma of p. In other words, by definition, it must be true that*

$$(f = \mathrm{f}(p)) \Rightarrow (\mathrm{c}(f) = \mathrm{c}(p))$$

Theorem 104 (Formula for $\mathrm{c}(f)$) *If f is the frequency of a pitch in the pitch system*

$$\psi = [\mu_\mathrm{c}, \mu_\mathrm{m}, f_0, p_{\mathrm{c},0}]$$

then

$$\mathrm{c}(f) = \mathrm{p_c}(f) \bmod \mu_\mathrm{c}$$

Proof

R1	Let		$f = \mathrm{f}(p)$
R2	71	$\Rightarrow$	$\mathrm{c}(p) = \mathrm{p_c}(p) \bmod \mu_\mathrm{c}$
R3	R1 & 98	$\Rightarrow$	$\mathrm{c}(p) = \mathrm{p_c}(f) \bmod \mu_\mathrm{c}$
R4	R1, R3 & 103	$\Rightarrow$	$\mathrm{c}(f) = \mathrm{p_c}(f) \bmod \mu_\mathrm{c}$

Deriving *MIPS* objects from a chromamorph

Definition 105 (Definition of $\mathrm{c}(q)$) *If q is the chromamorph of a pitch p in a pitch system ψ then the function $\mathrm{c}(q)$ must return the chroma of p. In other words, by definition, it must be true that*

$$(q = \mathrm{q}(p)) \Rightarrow (\mathrm{c}(q) = \mathrm{c}(p))$$

Theorem 106 (Formula for $\mathrm{c}(q)$) *If $q = [c, m]$ is the chromamorph of a pitch in a pitch system $\psi = [\mu_\mathrm{c}, \mu_\mathrm{m}, f_0, p_{\mathrm{c},0}]$ then*

$$\mathrm{c}(q) = c$$

Proof

R1	Let		$q = \mathrm{q}(p)$
R2	Let		$q = [c, m]$
R3	80	$\Rightarrow$	$\mathrm{q}(p) = [\mathrm{c}(p), \mathrm{m}(p)]$
R4	R1, R2 & R3	$\Rightarrow$	$\mathrm{c}(p) = c$
R5	R1, R4 & 105	$\Rightarrow$	$\mathrm{c}(q) = c$

Definition 107 (Definition of $\mathrm{m}(q)$) *If q is the chromamorph of a pitch p in a pitch system ψ then the function $\mathrm{m}(q)$ must return the morph of p. In other words, by definition, it must be true that*

$$(q = \mathrm{q}(p)) \Rightarrow (\mathrm{m}(q) = \mathrm{m}(p))$$

Theorem 108 (Formula for $\mathrm{m}(q)$) *If $q = [c, m]$ is the chromamorph of a pitch in a pitch system ψ then*

$$\mathrm{m}(q) = m$$

Proof

R1	Let		$q = \mathrm{q}(p)$
R2	Let		$q = [c, m]$
R3	80	$\Rightarrow$	$\mathrm{q}(p) = [\mathrm{c}(p), \mathrm{m}(p)]$
R4	R1, R2 & R3	$\Rightarrow$	$\mathrm{m}(p) = m$
R5	R1, R4 & 107	$\Rightarrow$	$\mathrm{m}(q) = m$

Theorem 109 ($q = [\mathrm{c}(q), \mathrm{m}(q)]$) *If q is a chromamorph in ψ then*

$$q = [\mathrm{c}(q), \mathrm{m}(q)]$$

Proof

R1	Let		$q = [c, m]$
R2	R1 & 106	$\Rightarrow$	$\mathrm{c}(q) = c$
R3	R1 & 108	$\Rightarrow$	$\mathrm{m}(q) = m$
R4	R1, R2 & R3	$\Rightarrow$	$q = [\mathrm{c}(q), \mathrm{m}(q)]$

Deriving *MIPS* objects from a chromatic genus

Definition 110 (Definition of $\mathrm{c}(g_{\mathrm{c}})$) *If g_{c} is the chromatic genus of a pitch p in a pitch system ψ then the function $\mathrm{c}(g_{\mathrm{c}})$ must return the chroma of p. In other words, by definition, it must be true that*

$$(g_{\mathrm{c}} = \mathrm{g_c}(p)) \Rightarrow (\mathrm{c}(g_{\mathrm{c}}) = \mathrm{c}(p))$$

Theorem 111 (Formula for $\mathrm{c}(g_{\mathrm{c}})$) *If g_{c} is the chromatic genus of a pitch in the pitch system*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

then

$$\mathrm{c}(g_{\mathrm{c}}) = g_{\mathrm{c}} \bmod \mu_{\mathrm{c}}$$

Proof

R1	Let		$g_{\mathrm{c}} = \mathrm{g_c}(p)$
R2	82	$\Rightarrow$	$\mathrm{g_c}(p) = \mathrm{p_c}(p) - \mu_{\mathrm{c}} \times \mathrm{o_m}(p)$
R3	R1 & R2	$\Rightarrow$	$g_{\mathrm{c}} = \mathrm{p_c}(p) - \mu_{\mathrm{c}} \times \mathrm{o_m}(p)$
R4	71	$\Rightarrow$	$\mathrm{c}(p) = \mathrm{p_c}(p) \bmod \mu_{\mathrm{c}}$
R5	R1, R4 & 110	$\Rightarrow$	$\mathrm{c}(g_{\mathrm{c}}) = \mathrm{p_c}(p) \bmod \mu_{\mathrm{c}}$
R6	70	$\Rightarrow$	$\mathrm{o_m}(p)$ is an integer
R7	R6 & 37	$\Rightarrow$	$(\mathrm{p_c}(p) - \mu_{\mathrm{c}} \times \mathrm{o_m}(p)) \bmod \mu_{\mathrm{c}} = \mathrm{p_c}(p) \bmod \mu_{\mathrm{c}}$
R8	R7 & R3	$\Rightarrow$	$g_{\mathrm{c}} \bmod \mu_{\mathrm{c}} = \mathrm{p_c}(p) \bmod \mu_{\mathrm{c}}$
R9	R5 & R8	$\Rightarrow$	$\mathrm{c}(g_{\mathrm{c}}) = g_{\mathrm{c}} \bmod \mu_{\mathrm{c}}$

Definition 112 (Definition of $\mathrm{d_o}(g_{\mathrm{c}})$) *If g_{c} is the chromatic genus of a pitch p in a pitch system ψ then the function $\mathrm{d_o}(g_{\mathrm{c}})$ must return the octave differenc of p. In other words, by definition, it must be true that*

$$(g_{\mathrm{c}} = \mathrm{g_c}(p)) \Rightarrow (\mathrm{d_o}(g_{\mathrm{c}}) = \mathrm{d_o}(p))$$

Theorem 113 (Formula for $\mathrm{d_o}(g_{\mathrm{c}})$) *If g_{c} is the chromatic genus of a pitch in the pitch system*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

then

$$\mathrm{d_o}(g_{\mathrm{c}}) = g_{\mathrm{c}} \operatorname{div} \mu_{\mathrm{c}}$$

Proof

R1	Let		$g_{\mathrm{c}} = \mathrm{g_c}(p)$
R2	82	$\Rightarrow$	$\mathrm{g_c}(p) = \mathrm{p_c}(p) - \mu_{\mathrm{c}} \times \mathrm{o_m}(p)$
R3	R1 & R2	$\Rightarrow$	$g_{\mathrm{c}} = \mathrm{p_c}(p) - \mu_{\mathrm{c}} \times \mathrm{o_m}(p)$
R4	81	$\Rightarrow$	$\mathrm{d_o}(p) = \mathrm{o_c}(p) - \mathrm{o_m}(p)$
R5	R1, R4 & 112	$\Rightarrow$	$\mathrm{d_o}(g_{\mathrm{c}}) = \mathrm{o_c}(p) - \mathrm{o_m}(p)$
R6	68	$\Rightarrow$	$\mathrm{o_c}(p) = \mathrm{p_c}(p) \operatorname{div} \mu_{\mathrm{c}}$
R7	R6 & R5	$\Rightarrow$	$\mathrm{d_o}(g_{\mathrm{c}}) = (\mathrm{p_c}(p) \operatorname{div} \mu_{\mathrm{c}}) - \mathrm{o_m}(p)$
R8	70	$\Rightarrow$	$\mathrm{o_m}(p)$ is an integer
R9	R8 & 50	$\Rightarrow$	$(\mathrm{p_c}(p) \operatorname{div} \mu_{\mathrm{c}}) - \mathrm{o_m}(p) = (\mathrm{p_c}(p) - \mu_{\mathrm{c}} \times \mathrm{o_m}(p)) \operatorname{div} \mu_{\mathrm{c}}$
R10	R9, R3 & R7	$\Rightarrow$	$\mathrm{d_o}(g_{\mathrm{c}}) = g_{\mathrm{c}} \operatorname{div} \mu_{\mathrm{c}}$

Deriving *MIPS* objects from a genus

Definition 114 (Chromatic genus of a genus) *If g is the genus of a pitch p in a pitch system ψ then the function $\mathrm{g_c}(g)$ must return the chromatic genus of p. In other words, by definition, it must be true that*

$$(g = \mathrm{g}(p)) \Rightarrow (\mathrm{g_c}(g) = \mathrm{g_c}(p))$$

Theorem 115 (Chromatic genus of a genus) *If $g = [g_{\mathrm{c}}, m]$ is the genus of a pitch in the pitch system ψ then*

$$\mathrm{g_c}(g) = g_{\mathrm{c}}$$

Proof

R1	Let		$g = [g_{\mathrm{c}}, m]$
R2	Let		$g = \mathrm{g}(p)$
R3	84	$\Rightarrow$	$\mathrm{g}(p) = [\mathrm{g_c}(p), \mathrm{m}(p)]$
R4	R2 & R3	$\Rightarrow$	$g = [\mathrm{g_c}(p), \mathrm{m}(p)]$
R5	R4 & R1	$\Rightarrow$	$g_{\mathrm{c}} = \mathrm{g_c}(p)$
R6	R5, R2 & 114	$\Rightarrow$	$\mathrm{g_c}(g) = g_{\mathrm{c}}$

Definition 116 (Morph of a genus) *If g is the genus of a pitch p in a pitch system ψ then the function* $\mathrm{m}(g)$ *must return the morph of p. In other words, by definition, it must be true that*

$$(g = \mathrm{g}(p)) \Rightarrow (\mathrm{m}(g) = \mathrm{m}(p))$$

Theorem 117 (Morph of a genus) *If $g = [g_\mathrm{c}, m]$ is the genus of a pitch in the pitch system ψ then*

$$\mathrm{m}(g) = m$$

Proof

R1	Let		$g = [g_\mathrm{c}, m]$
R2	Let		$g = \mathrm{g}(p)$
R3	84	$\Rightarrow$	$\mathrm{g}(p) = [\mathrm{g_c}(p), \mathrm{m}(p)]$
R4	R2 & R3	$\Rightarrow$	$g = [\mathrm{g_c}(p), \mathrm{m}(p)]$
R5	R4 & R1	$\Rightarrow$	$m = \mathrm{m}(p)$
R6	R5, R2 & 116	$\Rightarrow$	$\mathrm{m}(g) = m$

Theorem 118 *If g is a genus in a pitch system ψ then*

$$g = [\mathrm{g_c}(g), \mathrm{m}(g)]$$

Proof

R1	Let		$g = [g_\mathrm{c}, m]$
R2	R1 & 117	$\Rightarrow$	$\mathrm{m}(g) = m$
R3	R1 & 115	$\Rightarrow$	$\mathrm{g_c}(g) = g_\mathrm{c}$
R4	R1, R2 & R3	$\Rightarrow$	$g = [\mathrm{g_c}(g), \mathrm{m}(g)]$

Definition 119 (Chroma of a genus) *If g is the genus of a pitch p in a pitch system ψ then the function* $\mathrm{c}(g)$ *must return the chroma of p. In other words, by definition, it must be true that*

$$(g = \mathrm{g}(p)) \Rightarrow (\mathrm{c}(g) = \mathrm{c}(p))$$

Theorem 120 (Chroma of a genus) *If g is the genus of a pitch in the pitch system*

$$\psi = [\mu_\mathrm{c}, \mu_\mathrm{m}, f_0, p_{\mathrm{c},0}]$$

then

$$\mathrm{c}(g) = \mathrm{g_c}(g) \bmod \mu_\mathrm{c}$$

Proof

R1	Let		$g = \mathrm{g}(p)$
R2	84	$\Rightarrow$	$\mathrm{g}(p) = [\mathrm{g_c}(p), \mathrm{m}(p)]$
R3	R1 & R2	$\Rightarrow$	$g = [\mathrm{g_c}(p), \mathrm{m}(p)]$
R4	71	$\Rightarrow$	$\mathrm{c}(p) = \mathrm{p_c}(p) \bmod \mu_\mathrm{c}$
R5	R1 & 119	$\Rightarrow$	$\mathrm{c}(g) = \mathrm{p_c}(p) \bmod \mu_\mathrm{c}$
R6	82	$\Rightarrow$	$\mathrm{g_c}(p) = \mathrm{p_c}(p) - \mu_\mathrm{c} \times \mathrm{o_m}(p)$
R7	R6, R1 & 114	$\Rightarrow$	$\mathrm{g_c}(g) = \mathrm{p_c}(p) - \mu_\mathrm{c} \times \mathrm{o_m}(p)$
R8	70	$\Rightarrow$	$\mathrm{o_m}(p)$ is an integer
R9	R8 & 37	$\Rightarrow$	$(\mathrm{p_c}(p) - \mu_\mathrm{c} \times \mathrm{o_m}(p)) \bmod \mu_\mathrm{c} = \mathrm{p_c}(p) \bmod \mu_\mathrm{c}$
R10	R9 & R5	$\Rightarrow$	$\mathrm{c}(g) = (\mathrm{p_c}(p) - \mu_\mathrm{c} \times \mathrm{o_m}(p)) \bmod \mu_\mathrm{c}$
R11	R10 & R7	$\Rightarrow$	$\mathrm{c}(g) = \mathrm{g_c}(g) \bmod \mu_\mathrm{c}$

Definition 121 (Chromamorph of a genus) *If g is the genus of a pitch p in a pitch system ψ then the function $\mathrm{q}(g)$ must return the chromamorph of p. In other words, by definition, it must be true that*

$$(g = \mathrm{g}(p)) \Rightarrow (\mathrm{q}(g) = \mathrm{q}(p))$$

Theorem 122 (Chromamorph of a genus) *If g is the genus of a pitch in the pitch system ψ then*

$$\mathrm{q}(g) = [\mathrm{c}(g), \mathrm{m}(g)]$$

Proof

R1	Let		$g = \mathrm{g}(p)$
R2	R1 & 121	$\Rightarrow$	$\mathrm{q}(g) = \mathrm{q}(p)$
R3	80	$\Rightarrow$	$\mathrm{q}(p) = [\mathrm{c}(p), \mathrm{m}(p)]$
R4	R2 & R3	$\Rightarrow$	$\mathrm{q}(g) = [\mathrm{c}(p), \mathrm{m}(p)]$
R5	R4, R1 & 119	$\Rightarrow$	$\mathrm{q}(g) = [\mathrm{c}(g), \mathrm{m}(p)]$
R6	R5, R1 & 116	$\Rightarrow$	$\mathrm{q}(g) = [\mathrm{c}(g), \mathrm{m}(g)]$

Definition 123 (Definition of $\mathrm{d_o}(g)$) *If g is the genus of a pitch p in a pitch system ψ then the function $\mathrm{d_o}(g)$ must return the octave difference of p. In other words, by definition, it must be true that*

$$(g = \mathrm{g}(p)) \Rightarrow (\mathrm{d_o}(g) = \mathrm{d_o}(p))$$

Theorem 124 (Formula for $\mathrm{d_o}(g)$) *If g is the genus of a pitch in the pitch system*

$$\psi = [\mu_\mathrm{c}, \mu_\mathrm{m}, f_0, p_{\mathrm{c},0}]$$

then

$$\mathrm{d_o}(g) = \mathrm{g_c}(g) \text{ div } \mu_\mathrm{c}$$

Proof

R1	Let		$g = \mathrm{g}(p)$
R2	81	$\Rightarrow$	$\mathrm{d_o}(p) = \mathrm{o_c}(p) - \mathrm{o_m}(p)$
R3	R1, R2 & 123	$\Rightarrow$	$\mathrm{d_o}(g) = \mathrm{o_c}(p) - \mathrm{o_m}(p)$
R4	68	$\Rightarrow$	$\mathrm{o_c}(p) = \mathrm{p_c}(p) \text{ div } \mu_\mathrm{c}$
R5	R3 & R4	$\Rightarrow$	$\mathrm{d_o}(g) = (\mathrm{p_c}(p) \text{ div } \mu_\mathrm{c}) - \mathrm{o_m}(p)$
R6	82	$\Rightarrow$	$\mathrm{g_c}(p) = \mathrm{p_c}(p) - \mu_\mathrm{c} \times \mathrm{o_m}(p)$
R7	70	$\Rightarrow$	$\mathrm{o_m}(p)$ is an integer
R8	R7 & 50	$\Rightarrow$	$(\mathrm{p_c}(p) \text{ div } \mu_\mathrm{c}) - \mathrm{o_m}(p) = (\mathrm{p_c}(p) - \mu_\mathrm{c} \times \mathrm{o_m}(p)) \text{ div } \mu_\mathrm{c}$
R9	R8 & R6	$\Rightarrow$	$(\mathrm{p_c}(p) \text{ div } \mu_\mathrm{c}) - \mathrm{o_m}(p) = \mathrm{g_c}(p) \text{ div } \mu_\mathrm{c}$
R10	R9 & R5	$\Rightarrow$	$\mathrm{d_o}(g) = \mathrm{g_c}(p) \text{ div } \mu_\mathrm{c}$
R11	R10, R1 & 114	$\Rightarrow$	$\mathrm{d_o}(g) = \mathrm{g_c}(g) \text{ div } \mu_\mathrm{c}$

4.3.3 Equivalence relations between *MIPS* objects

Equivalence relations between pitches

Definition 125 (Chromatic pitch equivalence of pitches) *Two pitches p_1 and p_2 in a well-formed pitch system are* chromatic pitch equivalent *if and only if*

$$\mathrm{p_c}(p_1) = \mathrm{p_c}(p_2)$$

The fact that two pitches are chromatic pitch equivalent will be denoted

$$p_1 \equiv_{\mathrm{p_c}} p_2$$

Definition 126 (Morphetic pitch equivalence of pitches) *Two pitches p_1 and p_2 in a well-formed pitch system are* morphetic pitch equivalent *if and only if*

$$\mathrm{p_m}(p_1) = \mathrm{p_m}(p_2)$$

The fact that two pitches are morphetic pitch equivalent will be denoted

$$p_1 \equiv_{\mathrm{p_m}} p_2$$

Definition 127 (Frequency equivalence of pitches) *Two pitches p_1 and p_2 in a well-formed pitch system are* frequency equivalent *if and only if*

$$\mathrm{f}(p_1) = \mathrm{f}(p_2)$$

The fact that two pitches are frequency equivalent will be denoted

$$p_1 \equiv_{\mathrm{f}} p_2$$

Definition 128 (Chromatic octave equivalence of pitches) *Two pitches p_1 and p_2 in a well-formed pitch system are* chromatic octave equivalent *if and only if*

$$\mathrm{o_c}(p_1) = \mathrm{o_c}(p_2)$$

The fact that two pitches are chromatic octave equivalent will be denoted

$$p_1 \equiv_{\mathrm{o_c}} p_2$$

Definition 129 (Morphetic octave equivalence of pitches) *Two pitches p_1 and p_2 in a well-formed pitch system are* morphetic octave equivalent *if and only if*

$$\mathrm{o_m}(p_1) = \mathrm{o_m}(p_2)$$

The fact that two pitches are morphetic octave equivalent will be denoted

$$p_1 \equiv_{\mathrm{o_m}} p_2$$

Definition 130 (Chroma equivalence of pitches) *Two pitches p_1 and p_2 in a well-formed pitch system are* chroma equivalent *if and only if*

$$\mathrm{c}(p_1) = \mathrm{c}(p_2)$$

The fact that two pitches are chroma equivalent will be denoted

$$p_1 \equiv_{\mathrm{c}} p_2$$

Definition 131 (Morph equivalence of pitches) *Two pitches p_1 and p_2 in a well-formed pitch system are* morph equivalent *if and only if*

$$\mathrm{m}(p_1) = \mathrm{m}(p_2)$$

The fact that two pitches are morph equivalent will be denoted

$$p_1 \equiv_{\mathrm{m}} p_2$$

Definition 132 (Chromamorph equivalence of pitches) *Two pitches p_1 and p_2 in a well-formed pitch system are* chromamorph equivalent *if and only if*

$$\mathrm{q}(p_1) = \mathrm{q}(p_2)$$

The fact that two pitches are chromamorph equivalent will be denoted

$$p_1 \equiv_{\mathrm{q}} p_2$$

Definition 133 (Octave difference equivalence of pitches) *Two pitches p_1 and p_2 in a well-formed pitch system are* octave difference equivalent *if and only if*

$$\mathrm{d_o}(p_1) = \mathrm{d_o}(p_2)$$

The fact that two pitches are octave difference equivalent will be denoted

$$p_1 \equiv_{\mathrm{d_o}} p_2$$

Definition 134 (Chromatic genus equivalence of pitches) *Two pitches p_1 and p_2 in a well-formed pitch system are* chromatic genus equivalent *if and only if*

$$\mathrm{g_c}(p_1) = \mathrm{g_c}(p_2)$$

The fact that two pitches are chromatic genus equivalent will be denoted

$$p_1 \equiv_{\mathrm{g_c}} p_2$$

Definition 135 (Genus equivalence of pitches) *Two pitches p_1 and p_2 in a well-formed pitch system are* genus equivalent *if and only if*

$$\mathrm{g}(p_1) = \mathrm{g}(p_2)$$

The fact that two pitches are genus equivalent will be denoted

$$p_1 \equiv_{\mathrm{g}} p_2$$

Equivalence relations between chromatic pitches

Definition 136 ($p_{\mathrm{c},1} \equiv_{\mathrm{f}} p_{\mathrm{c},2}$) *Two chromatic pitches $p_{\mathrm{c},1}$ and $p_{\mathrm{c},2}$ in a well-formed pitch system are* frequency equivalent *if and only if*

$$\mathrm{f}(p_{\mathrm{c},1}) = \mathrm{f}(p_{\mathrm{c},2})$$

The fact that two chromatic pitches are frequency equivalent will be denoted

$$p_{\mathrm{c},1} \equiv_{\mathrm{f}} p_{\mathrm{c},2}$$

Definition 137 ($p_{\mathrm{c},1} \equiv_{\mathrm{o_c}} p_{\mathrm{c},2}$) *Two chromatic pitches $p_{\mathrm{c},1}$ and $p_{\mathrm{c},2}$ in a well-formed pitch system are* chromatic octave equivalent *if and only if*

$$\mathrm{o_c}(p_{\mathrm{c},1}) = \mathrm{o_c}(p_{\mathrm{c},2})$$

The fact that two chromatic pitches are chromatic octave equivalent will be denoted

$$p_{\mathrm{c},1} \equiv_{\mathrm{o_c}} p_{\mathrm{c},2}$$

Definition 138 ($p_{\mathrm{c},1} \equiv_{\mathrm{c}} p_{\mathrm{c},2}$) *Two chromatic pitches $p_{\mathrm{c},1}$ and $p_{\mathrm{c},2}$ in a well-formed pitch system are* chroma equivalent *if and only if*

$$\mathrm{c}(p_{\mathrm{c},1}) = \mathrm{c}(p_{\mathrm{c},2})$$

The fact that two chromatic pitches are chroma equivalent will be denoted

$$p_{\mathrm{c},1} \equiv_{\mathrm{c}} p_{\mathrm{c},2}$$

Equivalence relations between morphetic pitches

Definition 139 ($p_{\mathrm{m},1} \equiv_{\mathrm{o_m}} p_{\mathrm{m},2}$) *Two morphetic pitches* $p_{\mathrm{m},1}$ *and* $p_{\mathrm{m},2}$ *in a well-formed pitch system are* morphetic octave equivalent *if and only if*

$$\mathrm{o_m}\left(p_{\mathrm{m},1}\right) = \mathrm{o_m}\left(p_{\mathrm{m},2}\right)$$

The fact that two morphetic pitches are morphetic octave equivalent will be denoted

$$p_{\mathrm{m},1} \equiv_{\mathrm{o_m}} p_{\mathrm{m},2}$$

Definition 140 ($p_{\mathrm{m},1} \equiv_{\mathrm{m}} p_{\mathrm{m},2}$) *Two morphetic pitches* $p_{\mathrm{m},1}$ *and* $p_{\mathrm{m},2}$ *in a well-formed pitch system are* morph equivalent *if and only if*

$$\mathrm{m}\left(p_{\mathrm{m},1}\right) = \mathrm{m}\left(p_{\mathrm{m},2}\right)$$

The fact that two morphetic pitches are morph equivalent will be denoted

$$p_{\mathrm{m},1} \equiv_{\mathrm{m}} p_{\mathrm{m},2}$$

Equivalence relations between frequencies

Definition 141 ($f_1 \equiv_{\mathrm{p_c}} f_2$) *Two frequencies* f_1 *and* f_2 *in a well-formed pitch system are* chromatic pitch equivalent *if and only if*

$$\mathrm{p_c}\left(f_1\right) = \mathrm{p_c}\left(f_2\right)$$

The fact that two frequencies are chromatic pitch equivalent will be denoted

$$f_1 \equiv_{\mathrm{p_c}} f_2$$

Definition 142 ($f_1 \equiv_{\mathrm{o_c}} f_2$) *Two frequencies* f_1 *and* f_2 *in a well-formed pitch system are* chromatic octave equivalent *if and only if*

$$\mathrm{o_c}\left(f_1\right) = \mathrm{o_c}\left(f_2\right)$$

The fact that two frequencies are chromatic octave equivalent will be denoted

$$f_1 \equiv_{\mathrm{o_c}} f_2$$

Definition 143 ($f_1 \equiv_{\mathrm{c}} f_2$) *Two frequencies* f_1 *and* f_2 *in a well-formed pitch system are* chroma equivalent *if and only if*

$$\mathrm{c}\left(f_1\right) = \mathrm{c}\left(f_2\right)$$

The fact that two frequencies are chroma equivalent will be denoted

$$f_1 \equiv_{\mathrm{c}} f_2$$

Equivalence relations between chromamorphs

Definition 144 ($q_1 \equiv_{\mathrm{c}} q_2$) *Two chromamorphs* q_1 *and* q_2 *in a well-formed pitch system are* chroma equivalent *if and only if*

$$\mathrm{c}\left(q_1\right) = \mathrm{c}\left(q_2\right)$$

The fact that two chromamorphs are chroma equivalent will be denoted

$$q_1 \equiv_{\mathrm{c}} q_2$$

Definition 145 **($q_1 \equiv_{\mathrm{m}} q_2$)** *Two chromamorphs q_1 and q_2 in a well-formed pitch system are* morph equivalent *if and only if*

$$\mathrm{m}\left(q_1\right) = \mathrm{m}\left(q_2\right)$$

The fact that two chromamorphs are morph equivalent will be denoted

$$q_1 \equiv_{\mathrm{m}} q_2$$

Equivalence relations between chromatic genera

Definition 146 **($g_{\mathrm{c},1} \equiv_{\mathrm{c}} g_{\mathrm{c},2}$)** *Two chromatic genera $g_{\mathrm{c},1}$ and $g_{\mathrm{c},2}$ in a well-formed pitch system are* chroma equivalent *if and only if*

$$\mathrm{c}\left(g_{\mathrm{c},1}\right) = \mathrm{c}\left(g_{\mathrm{c},2}\right)$$

The fact that two chromatic genera are chroma equivalent will be denoted

$$g_{\mathrm{c},1} \equiv_{\mathrm{c}} g_{\mathrm{c},2}$$

Definition 147 **($g_{\mathrm{c},1} \equiv_{\mathrm{d_o}} g_{\mathrm{c},2}$)** *Two chromatic genera $g_{\mathrm{c},1}$ and $g_{\mathrm{c},2}$ in a well-formed pitch system are* octave difference equivalent *if and only if*

$$\mathrm{d_o}\left(g_{\mathrm{c},1}\right) = \mathrm{d_o}\left(g_{\mathrm{c},2}\right)$$

The fact that two chromatic genera are octave difference equivalent will be denoted

$$g_{\mathrm{c},1} \equiv_{\mathrm{d_o}} g_{\mathrm{c},2}$$

Equivalence relations between genera

Definition 148 **($g_1 \equiv_{\mathrm{g_c}} g_2$)** *Two genera g_1 and g_2 in a well-formed pitch system are* chromatic genus equivalent *if and only if*

$$\mathrm{g_c}\left(g_1\right) = \mathrm{g_c}\left(g_2\right)$$

The fact that two genera are chromatic genus equivalent will be denoted

$$g_1 \equiv_{\mathrm{g_c}} g_2$$

Definition 149 **($g_1 \equiv_{\mathrm{m}} g_2$)** *Two genera g_1 and g_2 in a well-formed pitch system are* morph equivalent *if and only if*

$$\mathrm{m}\left(g_1\right) = \mathrm{m}\left(g_2\right)$$

The fact that two genera are morph equivalent will be denoted

$$g_1 \equiv_{\mathrm{m}} g_2$$

Definition 150 **($g_1 \equiv_{\mathrm{c}} g_2$)** *Two genera g_1 and g_2 in a well-formed pitch system are* chroma equivalent *if and only if*

$$\mathrm{c}\left(g_1\right) = \mathrm{c}\left(g_2\right)$$

The fact that two genera are chroma equivalent will be denoted

$$g_1 \equiv_{\mathrm{c}} g_2$$

Definition 151 ($g_1 \equiv_{\mathrm{q}} g_2$) *Two genera g_1 and g_2 in a well-formed pitch system are* chromamorph equivalent *if and only if*

$$\mathrm{q}(g_1) = \mathrm{q}(g_2)$$

The fact that two genera are chromamorph equivalent will be denoted

$$g_1 \equiv_{\mathrm{q}} g_2$$

Definition 152 ($g_1 \equiv_{\mathrm{d_o}} g_2$) *Two genera g_1 and g_2 in a well-formed pitch system are* octave difference equivalent *if and only if*

$$\mathrm{d_o}(g_1) = \mathrm{d_o}(g_2)$$

The fact that two genera are octave difference equivalent will be denoted

$$g_1 \equiv_{\mathrm{d_o}} g_2$$

4.3.4 Inequalities between *MIPS* objects

Inequalities between two pitches

Definition 153 *If p_1 and p_2 are any two pitches in a pitch system ψ then p_1 is* chromatic pitch less than *p_2, denoted*

$$p_1 <_{\mathrm{p_c}} p_2$$

if and only if

$$\mathrm{p_c}(p_1) < \mathrm{p_c}(p_2)$$

Definition 154 *If p_1 and p_2 are any two pitches in a pitch system ψ then p_1 is* chromatic pitch less than or equal to *p_2, denoted*

$$p_1 \leq_{\mathrm{p_c}} p_2$$

if and only if

$$\mathrm{p_c}(p_1) \leq \mathrm{p_c}(p_2)$$

Definition 155 *If p_1 and p_2 are any two pitches in a pitch system ψ then p_1 is* chromatic pitch greater than *p_2, denoted*

$$p_1 >_{\mathrm{p_c}} p_2$$

if and only if

$$\mathrm{p_c}(p_1) > \mathrm{p_c}(p_2)$$

Definition 156 *If p_1 and p_2 are any two pitches in a pitch system ψ then p_1 is* chromatic pitch greater than or equal to *p_2, denoted*

$$p_1 \geq_{\mathrm{p_c}} p_2$$

if and only if

$$\mathrm{p_c}(p_1) \geq \mathrm{p_c}(p_2)$$

Definition 157 *If p_1 and p_2 are any two pitches in a pitch system ψ then p_1 is* morphetic pitch less than *p_2, denoted*

$$p_1 <_{\mathrm{p_m}} p_2$$

if and only if

$$\mathrm{p_m}(p_1) < \mathrm{p_m}(p_2)$$

Definition 158 *If* p_1 *and* p_2 *are any two pitches in a pitch system* ψ *then* p_1 *is* morphetic pitch less than or equal to p_2, *denoted*

$$p_1 \leq_{\mathrm{p_m}} p_2$$

if and only if

$$\mathrm{p_m}(p_1) \leq \mathrm{p_m}(p_2)$$

Definition 159 *If* p_1 *and* p_2 *are any two pitches in a pitch system* ψ *then* p_1 *is* morphetic pitch greater than p_2, *denoted*

$$p_1 >_{\mathrm{p_m}} p_2$$

if and only if

$$\mathrm{p_m}(p_1) > \mathrm{p_m}(p_2)$$

Definition 160 *If* p_1 *and* p_2 *are any two pitches in a pitch system* ψ *then* p_1 *is* morphetic pitch greater than or equal to p_2, *denoted*

$$p_1 \geq_{\mathrm{p_m}} p_2$$

if and only if

$$\mathrm{p_m}(p_1) \geq \mathrm{p_m}(p_2)$$

Definition 161 *If* p_1 *and* p_2 *are any two pitches in a pitch system* ψ *then* p_1 *is* frequency less than p_2, *denoted*

$$p_1 <_{\mathrm{f}} p_2$$

if and only if

$$\mathrm{f}(p_1) < \mathrm{f}(p_2)$$

Definition 162 *If* p_1 *and* p_2 *are any two pitches in a pitch system* ψ *then* p_1 *is* frequency less than or equal to p_2, *denoted*

$$p_1 \leq_{\mathrm{f}} p_2$$

if and only if

$$\mathrm{f}(p_1) \leq \mathrm{f}(p_2)$$

Definition 163 *If* p_1 *and* p_2 *are any two pitches in a pitch system* ψ *then* p_1 *is* frequency greater than p_2, *denoted*

$$p_1 >_{\mathrm{f}} p_2$$

if and only if

$$\mathrm{f}(p_1) > \mathrm{f}(p_2)$$

Definition 164 *If* p_1 *and* p_2 *are any two pitches in a pitch system* ψ *then* p_1 *is* frequency greater than or equal to p_2, *denoted*

$$p_1 \geq_{\mathrm{f}} p_2$$

if and only if

$$\mathrm{f}(p_1) \geq \mathrm{f}(p_2)$$

Definition 165 *If* p_1 *and* p_2 *are any two pitches in a pitch system* ψ *then* p_1 *is* chroma less than p_2, *denoted*

$$p_1 <_{\mathrm{c}} p_2$$

if and only if

$$\mathrm{c}(p_1) < \mathrm{c}(p_2)$$

Definition 166 *If* p_1 *and* p_2 *are any two pitches in a pitch system* ψ *then* p_1 *is* chroma less than or equal to p_2*, denoted*

$$p_1 \leq_{\mathrm{c}} p_2$$

if and only if

$$\mathrm{c}(p_1) \leq \mathrm{c}(p_2)$$

Definition 167 *If* p_1 *and* p_2 *are any two pitches in a pitch system* ψ *then* p_1 *is* chroma greater than p_2*, denoted*

$$p_1 >_{\mathrm{c}} p_2$$

if and only if

$$\mathrm{c}(p_1) > \mathrm{c}(p_2)$$

Definition 168 *If* p_1 *and* p_2 *are any two pitches in a pitch system* ψ *then* p_1 *is* chroma greater than or equal to p_2*, denoted*

$$p_1 \geq_{\mathrm{c}} p_2$$

if and only if

$$\mathrm{c}(p_1) \geq \mathrm{c}(p_2)$$

Definition 169 *If* p_1 *and* p_2 *are any two pitches in a pitch system* ψ *then* p_1 *is* morph less than p_2*, denoted*

$$p_1 <_{\mathrm{m}} p_2$$

if and only if

$$\mathrm{m}(p_1) < \mathrm{m}(p_2)$$

Definition 170 *If* p_1 *and* p_2 *are any two pitches in a pitch system* ψ *then* p_1 *is* morph less than or equal to p_2*, denoted*

$$p_1 \leq_{\mathrm{m}} p_2$$

if and only if

$$\mathrm{m}(p_1) \leq \mathrm{m}(p_2)$$

Definition 171 *If* p_1 *and* p_2 *are any two pitches in a pitch system* ψ *then* p_1 *is* morph greater than p_2*, denoted*

$$p_1 >_{\mathrm{m}} p_2$$

if and only if

$$\mathrm{m}(p_1) > \mathrm{m}(p_2)$$

Definition 172 *If* p_1 *and* p_2 *are any two pitches in a pitch system* ψ *then* p_1 *is* morph greater than or equal to p_2*, denoted*

$$p_1 \geq_{\mathrm{m}} p_2$$

if and only if

$$\mathrm{m}(p_1) \geq \mathrm{m}(p_2)$$

Definition 173 *If* p_1 *and* p_2 *are any two pitches in a pitch system* ψ *then* p_1 *is* chromatic genus less than p_2*, denoted*

$$p_1 <_{\mathrm{g_c}} p_2$$

if and only if

$$\mathrm{g_c}(p_1) < \mathrm{g_c}(p_2)$$

Definition 174 *If p_1 and p_2 are any two pitches in a pitch system ψ then p_1 is* chromatic genus less than or equal to p_2*, denoted*

$$p_1 \leq_{\mathrm{g_c}} p_2$$

if and only if

$$\mathrm{g_c}\left(p_1\right) \leq \mathrm{g_c}\left(p_2\right)$$

Definition 175 *If p_1 and p_2 are any two pitches in a pitch system ψ then p_1 is* chromatic genus greater than p_2*, denoted*

$$p_1 >_{\mathrm{g_c}} p_2$$

if and only if

$$\mathrm{g_c}\left(p_1\right) > \mathrm{g_c}\left(p_2\right)$$

Definition 176 *If p_1 and p_2 are any two pitches in a pitch system ψ then p_1 is* chromatic genus greater than or equal to p_2*, denoted*

$$p_1 \geq_{\mathrm{g_c}} p_2$$

if and only if

$$\mathrm{g_c}\left(p_1\right) \geq \mathrm{g_c}\left(p_2\right)$$

Inequalities between two chromatic pitches

Definition 177 *If $p_{\mathrm{c},1}$ and $p_{\mathrm{c},2}$ are any two chromatic pitches in a pitch system ψ then $p_{\mathrm{c},1}$ is* chroma less than $p_{\mathrm{c},2}$*, denoted*

$$p_{\mathrm{c},1} <_{\mathrm{c}} p_{\mathrm{c},2}$$

if and only if

$$\mathrm{c}\left(p_{\mathrm{c},1}\right) < \mathrm{c}\left(p_{\mathrm{c},2}\right)$$

Definition 178 *If $p_{\mathrm{c},1}$ and $p_{\mathrm{c},2}$ are any two chromatic pitches in a pitch system ψ then $p_{\mathrm{c},1}$ is* chroma less than or equal to $p_{\mathrm{c},2}$*, denoted*

$$p_{\mathrm{c},1} \leq_{\mathrm{c}} p_{\mathrm{c},2}$$

if and only if

$$\mathrm{c}\left(p_{\mathrm{c},1}\right) \leq \mathrm{c}\left(p_{\mathrm{c},2}\right)$$

Definition 179 *If $p_{\mathrm{c},1}$ and $p_{\mathrm{c},2}$ are any two chromatic pitches in a pitch system ψ then $p_{\mathrm{c},1}$ is* chroma greater than $p_{\mathrm{c},2}$*, denoted*

$$p_{\mathrm{c},1} >_{\mathrm{c}} p_{\mathrm{c},2}$$

if and only if

$$\mathrm{c}\left(p_{\mathrm{c},1}\right) > \mathrm{c}\left(p_{\mathrm{c},2}\right)$$

Definition 180 *If $p_{\mathrm{c},1}$ and $p_{\mathrm{c},2}$ are any two chromatic pitches in a pitch system ψ then $p_{\mathrm{c},1}$ is* chroma greater than or equal to $p_{\mathrm{c},2}$*, denoted*

$$p_{\mathrm{c},1} \geq_{\mathrm{c}} p_{\mathrm{c},2}$$

if and only if

$$\mathrm{c}\left(p_{\mathrm{c},1}\right) \geq \mathrm{c}\left(p_{\mathrm{c},2}\right)$$

Definition 181 *If* $p_{\mathrm{c},1}$ *and* $p_{\mathrm{c},2}$ *are any two chromatic pitches in a pitch system* ψ *then* $p_{\mathrm{c},1}$ *is* frequency less than $p_{\mathrm{c},2}$*, denoted*

$$p_{\mathrm{c},1} <_{\mathrm{f}} p_{\mathrm{c},2}$$

if and only if

$$\mathrm{f}\left(p_{\mathrm{c},1}\right) < \mathrm{f}\left(p_{\mathrm{c},2}\right)$$

Definition 182 *If* $p_{\mathrm{c},1}$ *and* $p_{\mathrm{c},2}$ *are any two chromatic pitches in a pitch system* ψ *then* $p_{\mathrm{c},1}$ *is* frequency less than or equal to $p_{\mathrm{c},2}$*, denoted*

$$p_{\mathrm{c},1} \leq_{\mathrm{f}} p_{\mathrm{c},2}$$

if and only if

$$\mathrm{f}\left(p_{\mathrm{c},1}\right) \leq \mathrm{f}\left(p_{\mathrm{c},2}\right)$$

Definition 183 *If* $p_{\mathrm{c},1}$ *and* $p_{\mathrm{c},2}$ *are any two chromatic pitches in a pitch system* ψ *then* $p_{\mathrm{c},1}$ *is* frequency greater than $p_{\mathrm{c},2}$*, denoted*

$$p_{\mathrm{c},1} >_{\mathrm{f}} p_{\mathrm{c},2}$$

if and only if

$$\mathrm{f}\left(p_{\mathrm{c},1}\right) > \mathrm{f}\left(p_{\mathrm{c},2}\right)$$

Definition 184 *If* $p_{\mathrm{c},1}$ *and* $p_{\mathrm{c},2}$ *are any two chromatic pitches in a pitch system* ψ *then* $p_{\mathrm{c},1}$ *is* frequency greater than or equal to $p_{\mathrm{c},2}$*, denoted*

$$p_{\mathrm{c},1} \geq_{\mathrm{f}} p_{\mathrm{c},2}$$

if and only if

$$\mathrm{f}\left(p_{\mathrm{c},1}\right) \geq \mathrm{f}\left(p_{\mathrm{c},2}\right)$$

Inequalities between two morphetic pitches

Definition 185 *If* $p_{\mathrm{m},1}$ *and* $p_{\mathrm{m},2}$ *are any two morphetic pitches in a pitch system* ψ *then* $p_{\mathrm{m},1}$ *is* morph less than $p_{\mathrm{m},2}$*, denoted*

$$p_{\mathrm{m},1} <_{\mathrm{m}} p_{\mathrm{m},2}$$

if and only if

$$\mathrm{m}\left(p_{\mathrm{m},1}\right) < \mathrm{m}\left(p_{\mathrm{m},2}\right)$$

Definition 186 *If* $p_{\mathrm{m},1}$ *and* $p_{\mathrm{m},2}$ *are any two morphetic pitches in a pitch system* ψ *then* $p_{\mathrm{m},1}$ *is* morph less than or equal to $p_{\mathrm{m},2}$*, denoted*

$$p_{\mathrm{m},1} \leq_{\mathrm{m}} p_{\mathrm{m},2}$$

if and only if

$$\mathrm{m}\left(p_{\mathrm{m},1}\right) \leq \mathrm{m}\left(p_{\mathrm{m},2}\right)$$

Definition 187 *If* $p_{\mathrm{m},1}$ *and* $p_{\mathrm{m},2}$ *are any two morphetic pitches in a pitch system* ψ *then* $p_{\mathrm{m},1}$ *is* morph greater than $p_{\mathrm{m},2}$*, denoted*

$$p_{\mathrm{m},1} >_{\mathrm{m}} p_{\mathrm{m},2}$$

if and only if

$$\mathrm{m}\left(p_{\mathrm{m},1}\right) > \mathrm{m}\left(p_{\mathrm{m},2}\right)$$

Definition 188 *If* $p_{\mathrm{m},1}$ *and* $p_{\mathrm{m},2}$ *are any two morphetic pitches in a pitch system* ψ *then* $p_{\mathrm{m},1}$ *is* morph greater than or equal to $p_{\mathrm{m},2}$, *denoted*

$$p_{\mathrm{m},1} \geq_{\mathrm{m}} p_{\mathrm{m},2}$$

if and only if

$$\mathrm{m}\left(p_{\mathrm{m},1}\right) \geq \mathrm{m}\left(p_{\mathrm{m},2}\right)$$

Inequalities between two frequencies

Definition 189 *If* f_1 *and* f_2 *are any two frequencies in a pitch system* ψ *then* f_1 *is* chromatic pitch less than f_2, *denoted*

$$f_1 <_{\mathrm{p_c}} f_2$$

if and only if

$$\mathrm{p_c}\left(f_1\right) < \mathrm{p_c}\left(f_2\right)$$

Definition 190 *If* f_1 *and* f_2 *are any two frequencies in a pitch system* ψ *then* f_1 *is* chromatic pitch less than or equal to f_2, *denoted*

$$f_1 \leq_{\mathrm{p_c}} f_2$$

if and only if

$$\mathrm{p_c}\left(f_1\right) \leq \mathrm{p_c}\left(f_2\right)$$

Definition 191 *If* f_1 *and* f_2 *are any two frequencies in a pitch system* ψ *then* f_1 *is* chromatic pitch greater than f_2, *denoted*

$$f_1 >_{\mathrm{p_c}} f_2$$

if and only if

$$\mathrm{p_c}\left(f_1\right) > \mathrm{p_c}\left(f_2\right)$$

Definition 192 *If* f_1 *and* f_2 *are any two frequencies in a pitch system* ψ *then* f_1 *is* chromatic pitch greater than or equal to f_2, *denoted*

$$f_1 \geq_{\mathrm{p_c}} f_2$$

if and only if

$$\mathrm{p_c}\left(f_1\right) \geq \mathrm{p_c}\left(f_2\right)$$

Definition 193 *If* f_1 *and* f_2 *are any two frequencies in a pitch system* ψ *then* f_1 *is* chroma less than f_2, *denoted*

$$f_1 <_{\mathrm{c}} f_2$$

if and only if

$$\mathrm{c}\left(f_1\right) < \mathrm{c}\left(f_2\right)$$

Definition 194 *If* f_1 *and* f_2 *are any two frequencies in a pitch system* ψ *then* f_1 *is* chroma less than or equal to f_2, *denoted*

$$f_1 \leq_{\mathrm{c}} f_2$$

if and only if

$$\mathrm{c}\left(f_1\right) \leq \mathrm{c}\left(f_2\right)$$

Definition 195 *If f_1 and f_2 are any two frequencies in a pitch system ψ then f_1 is* chroma greater than f_2*, denoted*

$$f_1 >_{\mathrm{c}} f_2$$

if and only if

$$\mathrm{c}\left(f_1\right) > \mathrm{c}\left(f_2\right)$$

Definition 196 *If f_1 and f_2 are any two frequencies in a pitch system ψ then f_1 is* chroma greater than or equal to f_2*, denoted*

$$f_1 \geq_{\mathrm{c}} f_2$$

if and only if

$$\mathrm{c}\left(f_1\right) \geq \mathrm{c}\left(f_2\right)$$

Inequalities between two chromatic genera

Definition 197 *If $g_{\mathrm{c},1}$ and $g_{\mathrm{c},2}$ are any two chromatic genera in a pitch system ψ then $g_{\mathrm{c},1}$ is* chroma less than $g_{\mathrm{c},2}$*, denoted*

$$g_{\mathrm{c},1} <_{\mathrm{c}} g_{\mathrm{c},2}$$

if and only if

$$\mathrm{c}\left(g_{\mathrm{c},1}\right) < \mathrm{c}\left(g_{\mathrm{c},2}\right)$$

Definition 198 *If $g_{\mathrm{c},1}$ and $g_{\mathrm{c},2}$ are any two chromatic genera in a pitch system ψ then $g_{\mathrm{c},1}$ is* chroma less than or equal to $g_{\mathrm{c},2}$*, denoted*

$$g_{\mathrm{c},1} \leq_{\mathrm{c}} g_{\mathrm{c},2}$$

if and only if

$$\mathrm{c}\left(g_{\mathrm{c},1}\right) \leq \mathrm{c}\left(g_{\mathrm{c},2}\right)$$

Definition 199 *If $g_{\mathrm{c},1}$ and $g_{\mathrm{c},2}$ are any two chromatic genera in a pitch system ψ then $g_{\mathrm{c},1}$ is* chroma greater than $g_{\mathrm{c},2}$*, denoted*

$$g_{\mathrm{c},1} >_{\mathrm{c}} g_{\mathrm{c},2}$$

if and only if

$$\mathrm{c}\left(g_{\mathrm{c},1}\right) > \mathrm{c}\left(g_{\mathrm{c},2}\right)$$

Definition 200 *If $g_{\mathrm{c},1}$ and $g_{\mathrm{c},2}$ are any two chromatic genera in a pitch system ψ then $g_{\mathrm{c},1}$ is* chroma greater than or equal to $g_{\mathrm{c},2}$*, denoted*

$$g_{\mathrm{c},1} \geq_{\mathrm{c}} g_{\mathrm{c},2}$$

if and only if

$$\mathrm{c}\left(g_{\mathrm{c},1}\right) \geq \mathrm{c}\left(g_{\mathrm{c},2}\right)$$

Inequalities between two genera

Definition 201 *If g_1 and g_2 are any two genera in a pitch system ψ then g_1 is* chromatic genus less than g_2*, denoted*

$$g_1 <_{\mathrm{g_c}} g_2$$

if and only if

$$\mathrm{g_c}\left(g_1\right) < \mathrm{g_c}\left(g_2\right)$$

Definition 202 *If g_1 and g_2 are any two genera in a pitch system ψ then g_1 is* chromatic genus less than or equal to g_2, *denoted*

$$g_1 \leq_{\mathrm{g_c}} g_2$$

if and only if

$$\mathrm{g_c}\left(g_1\right) \leq \mathrm{g_c}\left(g_2\right)$$

Definition 203 *If g_1 and g_2 are any two genera in a pitch system ψ then g_1 is* chromatic genus greater than g_2, *denoted*

$$g_1 >_{\mathrm{g_c}} g_2$$

if and only if

$$\mathrm{g_c}\left(g_1\right) > \mathrm{g_c}\left(g_2\right)$$

Definition 204 *If g_1 and g_2 are any two genera in a pitch system ψ then g_1 is* chromatic genus greater than or equal to g_2, *denoted*

$$g_1 \geq_{\mathrm{g_c}} g_2$$

if and only if

$$\mathrm{g_c}\left(g_1\right) \geq \mathrm{g_c}\left(g_2\right)$$

Definition 205 *If g_1 and g_2 are any two genera in a pitch system ψ then g_1 is* morph less than g_2, *denoted*

$$g_1 <_{\mathrm{m}} g_2$$

if and only if

$$\mathrm{m}\left(g_1\right) < \mathrm{m}\left(g_2\right)$$

Definition 206 *If g_1 and g_2 are any two genera in a pitch system ψ then g_1 is* morph less than or equal to g_2, *denoted*

$$g_1 \leq_{\mathrm{m}} g_2$$

if and only if

$$\mathrm{m}\left(g_1\right) \leq \mathrm{m}\left(g_2\right)$$

Definition 207 *If g_1 and g_2 are any two genera in a pitch system ψ then g_1 is* morph greater than g_2, *denoted*

$$g_1 >_{\mathrm{m}} g_2$$

if and only if

$$\mathrm{m}\left(g_1\right) > \mathrm{m}\left(g_2\right)$$

Definition 208 *If g_1 and g_2 are any two genera in a pitch system ψ then g_1 is* morph greater than or equal to g_2, *denoted*

$$g_1 \geq_{\mathrm{m}} g_2$$

if and only if

$$\mathrm{m}\left(g_1\right) \geq \mathrm{m}\left(g_2\right)$$

Definition 209 *If g_1 and g_2 are any two genera in a pitch system ψ then g_1 is* chroma less than g_2, *denoted*

$$g_1 <_{\mathrm{c}} g_2$$

if and only if

$$\mathrm{c}\left(g_1\right) < \mathrm{c}\left(g_2\right)$$

Definition 210 *If g_1 and g_2 are any two genera in a pitch system ψ then g_1 is* chroma less than or equal to g_2*, denoted*

$$g_1 \leq_{\mathrm{c}} g_2$$

if and only if

$$\mathrm{c}\,(g_1) \leq \mathrm{c}\,(g_2)$$

Definition 211 *If g_1 and g_2 are any two genera in a pitch system ψ then g_1 is* chroma greater than g_2*, denoted*

$$g_1 >_{\mathrm{c}} g_2$$

if and only if

$$\mathrm{c}\,(g_1) > \mathrm{c}\,(g_2)$$

Definition 212 *If g_1 and g_2 are any two genera in a pitch system ψ then g_1 is* chroma greater than or equal to g_2*, denoted*

$$g_1 \geq_{\mathrm{c}} g_2$$

if and only if

$$\mathrm{c}\,(g_1) \geq \mathrm{c}\,(g_2)$$

4.4 *MIPS* intervals

4.4.1 Intervals between two *MIPS* objects

Intervals between two chromae

Definition 213 ($\Delta\,\mathrm{c}\,(c_1, c_2)$) *If c_1 and c_2 are two chromae in a well-formed pitch system*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

then the chroma interval from c_1 to c_2 *is given by the following equation:*

$$\Delta\,\mathrm{c}\,(c_1, c_2) = (c_2 - c_1) \bmod \mu_{\mathrm{c}}$$

Theorem 214 *If $\Delta c = \Delta\,\mathrm{c}\,(c_1, c_2)$ where c_1 and c_2 are any two chromae in a pitch system*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

then Δc can only take any value such that

$$(0 \leq \Delta c < \mu_{\mathrm{c}}) \wedge (\Delta c \in \mathbb{Z})$$

where $\mathbb{Z}$ is the universal set of integers.

Proof

R1	Let		$\Delta c = \Delta\,\mathrm{c}\,(c_1, c_2)$ where c_1 and c_2 are any two chromae in ψ.
R2	72	$\Rightarrow$	c_1 and c_2 can only take any value such that $(0 \leq c_1, c_2 < \mu_\mathrm{c}) \wedge (c_1, c_2 \in \mathbb{Z})$
R3	R1 & 213	$\Rightarrow$	$\Delta c = (c_2 - c_1) \bmod \mu_\mathrm{c}$
R4	R3	$\Rightarrow$	$\Delta c = c_2 \bmod \mu_\mathrm{c}$ when $c_1 = 0$.
R5	61	$\Rightarrow$	μ_c can only take any positive integer value.
R6	R5, 44 & R4	$\Rightarrow$	$\Delta c = c_2$ when $c_1 = 0$.
R7	R6 & R2	$\Rightarrow$	Δc can take any value such that $(0 \leq \Delta c < \mu_\mathrm{c}) \wedge (\Delta c \in \mathbb{Z})$.
R8	R3 & 33	$\Rightarrow$	$\Delta c = (c_2 - c_1) - \mu_\mathrm{c} \times \mathrm{int}\left(\frac{c_2 - c_1}{\mu_\mathrm{c}}\right)$
R9	R8, 27, R5 & R2	$\Rightarrow$	Δc is an integer.
R10	41, R3 & R5	$\Rightarrow$	$0 \leq \Delta c < \mu_\mathrm{c}$
R11	R7, R9 & R10	$\Rightarrow$	Δc can only take any value such that $(0 \leq \Delta c < \mu_\mathrm{c}) \wedge (\Delta c \in \mathbb{Z})$

Theorem 215 *If*

$$\psi = [\mu_\mathrm{c}, \mu_\mathrm{m}, f_0, p_{\mathrm{c},0}]$$

is a pitch system and Δc *is a chroma interval in* ψ *then:*

$$\Delta c \bmod \mu_\mathrm{c} = \Delta c$$

Proof

R1	33	$\Rightarrow$	$\Delta c \bmod \mu_\mathrm{c} = \Delta c - \mu_\mathrm{c} \times \mathrm{int}\left(\frac{\Delta c}{\mu_\mathrm{c}}\right)$
R2	214	$\Rightarrow$	$\mathrm{int}\left(\frac{\Delta c}{\mu_\mathrm{c}}\right) = 0$
R3	R1 & R2	$\Rightarrow$	$\Delta c \bmod \mu_\mathrm{c} = \Delta c - \mu_\mathrm{c} \times 0 = \Delta c$

Theorem 216 *If*

$$\psi = [\mu_\mathrm{c}, \mu_\mathrm{m}, f_0, p_{\mathrm{c},0}]$$

is a pitch system and Δc *is a chroma interval in* ψ *then:*

$$\Delta c \operatorname{div} \mu_\mathrm{c} = 0$$

Proof

R1	48	$\Rightarrow$	$\Delta c \operatorname{div} \mu_{\mathrm{c}} = \operatorname{int}\left(\frac{\Delta c}{\mu_{\mathrm{c}}}\right)$
R2	214	$\Rightarrow$	$\operatorname{int}\left(\frac{\Delta c}{\mu_{\mathrm{c}}}\right) = 0$
R3	R1 & R2	$\Rightarrow$	$\Delta c \operatorname{div} \mu_{\mathrm{c}} = 0$

Intervals between two morphs

Definition 217 (Morph interval) *If m_1 and m_2 are two morphs in a well-formed pitch system*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

then the morph interval from m_1 to m_2 *is given by the following equation:*

$$\Delta\,\mathrm{m}\,(m_1, m_2) = (m_2 - m_1) \bmod \mu_{\mathrm{m}}$$

Theorem 218 *If $\Delta m = \Delta\,\mathrm{m}\,(m_1, m_2)$ where m_1 and m_2 are any two morphs in a pitch system*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

then Δm can only take any value such that

$$(0 \leq \Delta m < \mu_{\mathrm{m}}) \wedge (\Delta m \in \mathbb{Z})$$

where $\mathbb{Z}$ is the universal set of integers.

Proof

R1	Let		$\Delta m = \Delta\,\mathrm{m}\,(m_1, m_2)$ where m_1 and m_2 are any two morphs in ψ.
R2	77	$\Rightarrow$	m_1 and m_2 can only take any value such that $(0 \leq m_1, m_2 < \mu_\mathrm{m}) \wedge (m_1, m_2 \in \mathbb{Z})$
R3	R1 & 217	$\Rightarrow$	$\Delta m = (m_2 - m_1) \bmod \mu_\mathrm{m}$
R4	R3	$\Rightarrow$	$\Delta m = m_2 \bmod \mu_\mathrm{m}$ when $m_1 = 0$.
R5	61	$\Rightarrow$	μ_m can only take any positive integer value.
R6	R5, 44 & R4	$\Rightarrow$	$\Delta m = m_2$ when $m_1 = 0$.
R7	R6 & R2	$\Rightarrow$	Δm can take any value such that $(0 \leq \Delta m < \mu_\mathrm{m}) \wedge (\Delta m \in \mathbb{Z})$.
R8	R3 & 33	$\Rightarrow$	$\Delta m = (m_2 - m_1) - \mu_\mathrm{m} \times \mathrm{int}\left(\frac{m_2 - m_1}{\mu_\mathrm{m}}\right)$
R9	R8, 27, R5 & R2	$\Rightarrow$	Δm is an integer.
R10	41, R3 & R5	$\Rightarrow$	$0 \leq \Delta m < \mu_\mathrm{m}$
R11	R7, R9 & R10	$\Rightarrow$	Δm can only take any value such that $(0 \leq \Delta m < \mu_\mathrm{m}) \wedge (\Delta m \in \mathbb{Z})$

Theorem 219 *If*

$$\psi = [\mu_\mathrm{c}, \mu_\mathrm{m}, f_0, p_{\mathrm{c},0}]$$

is a pitch system and Δm is a morph interval in ψ then:

$$\Delta m \bmod \mu_\mathrm{m} = \Delta m$$

Proof

R1	33	$\Rightarrow$	$\Delta m \bmod \mu_\mathrm{m} = \Delta m - \mu_\mathrm{m} \times \mathrm{int}\left(\frac{\Delta m}{\mu_\mathrm{m}}\right)$
R2	218	$\Rightarrow$	$\mathrm{int}\left(\frac{\Delta m}{\mu_\mathrm{m}}\right) = 0$
R3	R1 & R2	$\Rightarrow$	$\Delta m \bmod \mu_\mathrm{m} = \Delta m - \mu_\mathrm{m} \times 0 = \Delta m$

Theorem 220 *If*

$$\psi = [\mu_\mathrm{c}, \mu_\mathrm{m}, f_0, p_{\mathrm{c},0}]$$

is a pitch system and Δm is a morph interval in ψ then:

$$\Delta m \ \mathrm{div}\ \mu_\mathrm{m} = 0$$

Proof

R1	48	$\Rightarrow$	$\Delta m \text{ div } \mu_{\text{m}} = \text{int}\left(\frac{\Delta m}{\mu_{\text{m}}}\right)$
R2	218	$\Rightarrow$	$\text{int}\left(\frac{\Delta m}{\mu_{\text{m}}}\right) = 0$
R3	R1 & R2	$\Rightarrow$	$\Delta m \text{ div } \mu_{\text{m}} = 0$

Intervals between two chromamorphs

Definition 221 (Definition of $\Delta\,\mathrm{c}\,(q_1, q_2)$) *If q_1 and q_2 are two chromamorphs in a pitch system ψ then the* chroma interval from q_1 to q_2 *is defined and denoted as follows:*

$$\Delta\,\mathrm{c}\,(q_1, q_2) = \Delta\,\mathrm{c}\,(\mathrm{c}\,(q_1)\,, \mathrm{c}\,(q_2))$$

Definition 222 (Definition of $\Delta\,\mathrm{m}\,(q_1, q_2)$) *If q_1 and q_2 are two chromamorphs in a pitch system ψ then the* morph interval from q_1 to q_2 *is defined and denoted as follows:*

$$\Delta\,\mathrm{m}\,(q_1, q_2) = \Delta\,\mathrm{m}\,(\mathrm{m}\,(q_1)\,, \mathrm{m}\,(q_2))$$

Definition 223 (Definition of $\Delta\,\mathrm{q}\,(q_1, q_2)$) *If q_1 and q_2 are two chromamorphs in a pitch system ψ then the* chromamorph interval from q_1 to q_2 *is defined and denoted as follows:*

$$\Delta\,\mathrm{q}\,(q_1, q_2) = [\Delta\,\mathrm{c}\,(q_1, q_2)\,, \Delta\,\mathrm{m}\,(q_1, q_2)]$$

Intervals between two chromatic genera

Definition 224 (Definition of $\Delta\,\mathrm{c}\,(g_{\mathrm{c},1}, g_{\mathrm{c},2})$) *If $g_{\mathrm{c},1}$ and $g_{\mathrm{c},2}$ are two chromatic genera in a pitch system ψ then the* chroma interval from $g_{\mathrm{c},1}$ to $g_{\mathrm{c},2}$ *is defined and denoted as follows:*

$$\Delta\,\mathrm{c}\,(g_{\mathrm{c},1}, g_{\mathrm{c},2}) = \Delta\,\mathrm{c}\,(\mathrm{c}\,(g_{\mathrm{c},1})\,, \mathrm{c}\,(g_{\mathrm{c},2}))$$

Theorem 225 (Formula for $\Delta\,\mathrm{c}\,(g_{\mathrm{c},1}, g_{\mathrm{c},2})$) *If $g_{\mathrm{c},1}$ and $g_{\mathrm{c},2}$ are two chromatic genera in a pitch system*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

then the chroma interval from $g_{\mathrm{c},1}$ to $g_{\mathrm{c},2}$ is given by the following expression:

$$\Delta\,\mathrm{c}\,(g_{\mathrm{c},1}, g_{\mathrm{c},2}) = (g_{\mathrm{c},2} - g_{\mathrm{c},1}) \bmod \mu_{\mathrm{c}}$$

Proof

R1	224	$\Rightarrow$	$\Delta\,\mathrm{c}\,(g_{\mathrm{c},1}, g_{\mathrm{c},2}) = \Delta\,\mathrm{c}\,(\mathrm{c}\,(g_{\mathrm{c},1})\,, \mathrm{c}\,(g_{\mathrm{c},2}))$
R2	111	$\Rightarrow$	$\mathrm{c}\,(g_{\mathrm{c},1}) = g_{\mathrm{c},1} \bmod \mu_{\mathrm{c}}$
R3	111	$\Rightarrow$	$\mathrm{c}\,(g_{\mathrm{c},2}) = g_{\mathrm{c},2} \bmod \mu_{\mathrm{c}}$
R4	213	$\Rightarrow$	$\Delta\,\mathrm{c}\,(\mathrm{c}\,(g_{\mathrm{c},1})\,, \mathrm{c}\,(g_{\mathrm{c},2})) = (\mathrm{c}\,(g_{\mathrm{c},2}) - \mathrm{c}\,(g_{\mathrm{c},1})) \bmod \mu_{\mathrm{c}}$
R5	R2, R3 & R4	$\Rightarrow$	$\Delta\,\mathrm{c}\,(\mathrm{c}\,(g_{\mathrm{c},1})\,, \mathrm{c}\,(g_{\mathrm{c},2})) = (g_{\mathrm{c},2} \bmod \mu_{\mathrm{c}} - g_{\mathrm{c},1} \bmod \mu_{\mathrm{c}}) \bmod \mu_{\mathrm{c}}$
R6	R5 & 38	$\Rightarrow$	$\Delta\,\mathrm{c}\,(\mathrm{c}\,(g_{\mathrm{c},1})\,, \mathrm{c}\,(g_{\mathrm{c},2})) = (g_{\mathrm{c},2} - g_{\mathrm{c},1} \bmod \mu_{\mathrm{c}}) \bmod \mu_{\mathrm{c}}$
R7	R6 & 38	$\Rightarrow$	$\Delta\,\mathrm{c}\,(\mathrm{c}\,(g_{\mathrm{c},1})\,, \mathrm{c}\,(g_{\mathrm{c},2})) = (g_{\mathrm{c},2} - g_{\mathrm{c},1}) \bmod \mu_{\mathrm{c}}$
R8	R7 & 224	$\Rightarrow$	$\Delta\,\mathrm{c}\,(g_{\mathrm{c},1}, g_{\mathrm{c},2}) = (g_{\mathrm{c},2} - g_{\mathrm{c},1}) \bmod \mu_{\mathrm{c}}$

Intervals between two genera

Definition 226 ($\Delta\,\mathrm{c}\,(g_1, g_2)$) *If g_1 and g_2 are two genera in a pitch system ψ then the* chroma interval from g_1 to g_2 *is defined and denoted as follows:*

$$\Delta\,\mathrm{c}\,(g_1, g_2) = \Delta\,\mathrm{c}\,(\mathrm{c}\,(g_1)\,, \mathrm{c}\,(g_2))$$

Theorem 227 (Formula for $\Delta\,\mathrm{c}\,(g_1, g_2)$) *If g_1 and g_2 are two genera in a pitch system*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

then the chroma interval from g_1 to g_2 is given by the following expression:

$$\Delta\,\mathrm{c}\,(g_1, g_2) = (\mathrm{g_c}\,(g_2) - \mathrm{g_c}\,(g_1)) \bmod \mu_{\mathrm{c}}$$

Proof

R1	226	$\Rightarrow$	$\Delta\,\mathrm{c}\,(g_1, g_2) = \Delta\,\mathrm{c}\,(\mathrm{c}\,(g_1)\,, \mathrm{c}\,(g_2))$
R2	120	$\Rightarrow$	$\mathrm{c}\,(g_1) = \mathrm{g_c}\,(g_1) \bmod \mu_\mathrm{c}$
R3	120	$\Rightarrow$	$\mathrm{c}\,(g_2) = \mathrm{g_c}\,(g_2) \bmod \mu_\mathrm{c}$
R4	213	$\Rightarrow$	$\Delta\,\mathrm{c}\,(\mathrm{c}\,(g_1)\,, \mathrm{c}\,(g_2)) = (\mathrm{c}\,(g_2) - \mathrm{c}\,(g_1)) \bmod \mu_\mathrm{c}$
R5	R2, R3 & R4	$\Rightarrow$	$\Delta\,\mathrm{c}\,(\mathrm{c}\,(g_1)\,, \mathrm{c}\,(g_2)) = (\mathrm{g_c}\,(g_2) \bmod \mu_\mathrm{c} - \mathrm{g_c}\,(g_1) \bmod \mu_\mathrm{c}) \bmod \mu_\mathrm{c}$
R6	R5 & 38	$\Rightarrow$	$\Delta\,\mathrm{c}\,(\mathrm{c}\,(g_1)\,, \mathrm{c}\,(g_2)) = (\mathrm{g_c}\,(g_2) - \mathrm{g_c}\,(g_1) \bmod \mu_\mathrm{c}) \bmod \mu_\mathrm{c}$
R7	R6 & 38	$\Rightarrow$	$\Delta\,\mathrm{c}\,(\mathrm{c}\,(g_1)\,, \mathrm{c}\,(g_2)) = (\mathrm{g_c}\,(g_2) - \mathrm{g_c}\,(g_1)) \bmod \mu_\mathrm{c}$
R8	R1 & R7	$\Rightarrow$	$\Delta\,\mathrm{c}\,(g_{\mathrm{c},1}, g_{\mathrm{c},2}) = (\mathrm{g_c}\,(g_2) - \mathrm{g_c}\,(g_1)) \bmod \mu_\mathrm{c}$

Definition 228 (Morph interval between two genera) *If g_1 and g_2 are two genera in a pitch system ψ then the* morph interval from g_1 to g_2 *is defined and denoted as follows:*

$$\Delta\,\mathrm{m}\,(g_1, g_2) = \Delta\,\mathrm{m}\,(\mathrm{m}\,(g_1)\,, \mathrm{m}\,(g_2))$$

Definition 229 ($\Delta\,\mathrm{q}\,(g_1, g_2)$) *If g_1 and g_2 are two genera in a pitch system ψ then the* chromamorph interval from g_1 to g_2 *is defined and denoted as follows:*

$$\Delta\,\mathrm{q}\,(g_1, g_2) = \Delta\,\mathrm{q}\,(\mathrm{q}\,(g_1)\,, \mathrm{q}\,(g_2))$$

Definition 230 (Chromatic genus interval between two genera) *If g_1 and g_2 are two genera in a pitch system*

$$\psi = [\mu_\mathrm{c}, \mu_\mathrm{m}, f_0, p_{\mathrm{c},0}]$$

then the chromatic genus interval from g_1 to g_2 *is defined and denoted as follows:*

$$\Delta\,\mathrm{g_c}\,(g_1, g_2) = \mathrm{g_c}\,(g_2) - \mathrm{g_c}\,(g_1) - \mu_\mathrm{c} \times ((\mathrm{m}\,(g_2) - \mathrm{m}\,(g_1)) \operatorname{div} \mu_\mathrm{m})$$

Definition 231 (Genus interval between two genera) *If g_1 and g_2 are two genera in a pitch system ψ then the* genus interval from g_1 to g_2 *is defined and denoted as follows:*

$$\Delta\,\mathrm{g}\,(g_1, g_2) = [\Delta\,\mathrm{g_c}\,(g_1, g_2)\,, \Delta\,\mathrm{m}\,(g_1, g_2)]$$

Intervals between two chromatic pitches

Definition 232 (Definition of $\Delta\,\mathrm{c}\,(p_{\mathrm{c},1}, p_{\mathrm{c},2})$) *If $p_{\mathrm{c},1}$ and $p_{\mathrm{c},2}$ are two chromatic pitches in a pitch system ψ then the* chroma interval from $p_{\mathrm{c},1}$ to $p_{\mathrm{c},2}$ *is defined and denoted as follows:*

$$\Delta\,\mathrm{c}\,(p_{\mathrm{c},1}, p_{\mathrm{c},2}) = \Delta\,\mathrm{c}\,(\mathrm{c}\,(p_{\mathrm{c},1})\,, \mathrm{c}\,(p_{\mathrm{c},2}))$$

Theorem 233 (Formula for $\Delta\,\mathrm{c}\,(p_{\mathrm{c},1}, p_{\mathrm{c},2})$) *If $p_{\mathrm{c},1}$ and $p_{\mathrm{c},2}$ are two chromatic pitches in a pitch system*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

then the chroma interval from $p_{\mathrm{c},1}$ to $p_{\mathrm{c},2}$ is given by:

$$\Delta\,\mathrm{c}\,(p_{\mathrm{c},1}, p_{\mathrm{c},2}) = (p_{\mathrm{c},2} - p_{\mathrm{c},1}) \bmod \mu_{\mathrm{c}}$$

Proof

R1	232	$\Rightarrow$	$\Delta\,\mathrm{c}\,(p_{\mathrm{c},1}, p_{\mathrm{c},2}) = \Delta\,\mathrm{c}\,(\mathrm{c}\,(p_{\mathrm{c},1}), \mathrm{c}\,(p_{\mathrm{c},2}))$
R2	R1 & 213	$\Rightarrow$	$\Delta\,\mathrm{c}\,(p_{\mathrm{c},1}, p_{\mathrm{c},2}) = (\mathrm{c}\,(p_{\mathrm{c},2}) - \mathrm{c}\,(p_{\mathrm{c},1})) \bmod \mu_{\mathrm{c}}$
R3	93	$\Rightarrow$	$\mathrm{c}\,(p_{\mathrm{c},1}) = p_{\mathrm{c},1} \bmod \mu_{\mathrm{c}}$
R4	93	$\Rightarrow$	$\mathrm{c}\,(p_{\mathrm{c},2}) = p_{\mathrm{c},2} \bmod \mu_{\mathrm{c}}$
R5	R2, R3 & R4	$\Rightarrow$	$\Delta\,\mathrm{c}\,(p_{\mathrm{c},1}, p_{\mathrm{c},2}) = (p_{\mathrm{c},2} \bmod \mu_{\mathrm{c}} - p_{\mathrm{c},1} \bmod \mu_{\mathrm{c}}) \bmod \mu_{\mathrm{c}}$
R6	R5 & 38	$\Rightarrow$	$\Delta\,\mathrm{c}\,(p_{\mathrm{c},1}, p_{\mathrm{c},2}) = (p_{\mathrm{c},2} - p_{\mathrm{c},1} \bmod \mu_{\mathrm{c}}) \bmod \mu_{\mathrm{c}}$
R7	R6 & 38	$\Rightarrow$	$\Delta\,\mathrm{c}\,(p_{\mathrm{c},1}, p_{\mathrm{c},2}) = (p_{\mathrm{c},2} - p_{\mathrm{c},1}) \bmod \mu_{\mathrm{c}}$

Definition 234 (Definition of $\Delta\,\mathrm{f}\,(p_{\mathrm{c},1}, p_{\mathrm{c},2})$) *If $p_{\mathrm{c},1}$ and $p_{\mathrm{c},2}$ are two chromatic pitches in a pitch system ψ then the* frequency interval from $p_{\mathrm{c},1}$ to $p_{\mathrm{c},2}$ *is defined and denoted as follows:*

$$\Delta\,\mathrm{f}\,(p_{\mathrm{c},1}, p_{\mathrm{c},2}) = \Delta\,\mathrm{f}\,(\mathrm{f}\,(p_{\mathrm{c},1}), \mathrm{f}\,(p_{\mathrm{c},2}))$$

The function $\Delta\,\mathrm{f}\,(f_1, f_2)$ is defined in Definition 242 below.

Theorem 235 (Formula for $\Delta\,\mathrm{f}\,(p_{\mathrm{c},1}, p_{\mathrm{c},2})$) *If $p_{\mathrm{c},1}$ and $p_{\mathrm{c},2}$ are two chromatic pitches in a pitch system*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

then the frequency interval from $p_{\mathrm{c},1}$ to $p_{\mathrm{c},2}$ is given by the following formula:

$$\Delta\,\mathrm{f}\,(p_{\mathrm{c},1}, p_{\mathrm{c},2}) = 2^{(p_{\mathrm{c},2} - p_{\mathrm{c},1})/\mu_{\mathrm{c}}}$$

Proof

R1	234	$\Rightarrow$	$\Delta\,\mathrm{f}\left(p_{\mathrm{c},1}, p_{\mathrm{c},2}\right) = \Delta\,\mathrm{f}\left(\mathrm{f}\left(p_{\mathrm{c},1}\right), \mathrm{f}\left(p_{\mathrm{c},2}\right)\right)$
R2	242	$\Rightarrow$	$\Delta\,\mathrm{f}\left(\mathrm{f}\left(p_{\mathrm{c},1}\right), \mathrm{f}\left(p_{\mathrm{c},2}\right)\right) = \frac{\mathrm{f}(p_{\mathrm{c},2})}{\mathrm{f}(p_{\mathrm{c},1})}$
R3	89	$\Rightarrow$	$\mathrm{f}\left(p_{\mathrm{c},2}\right) = f_0 \times 2^{(p_{\mathrm{c},2}-p_{\mathrm{c},0})/\mu_{\mathrm{c}}}$
R4	89	$\Rightarrow$	$\mathrm{f}\left(p_{\mathrm{c},1}\right) = f_0 \times 2^{(p_{\mathrm{c},1}-p_{\mathrm{c},0})/\mu_{\mathrm{c}}}$
R5	R2, R3 & R4	$\Rightarrow$	$\Delta\,\mathrm{f}\left(\mathrm{f}\left(p_{\mathrm{c},1}\right), \mathrm{f}\left(p_{\mathrm{c},2}\right)\right) = \frac{f_0 \times 2^{(p_{\mathrm{c},2}-p_{\mathrm{c},0})/\mu_{\mathrm{c}}}}{f_0 \times 2^{(p_{\mathrm{c},1}-p_{\mathrm{c},0})/\mu_{\mathrm{c}}}}$

$$= \frac{2^{(p_{\mathrm{c},2}-p_{\mathrm{c},0})/\mu_{\mathrm{c}}}}{2^{(p_{\mathrm{c},1}-p_{\mathrm{c},0})/\mu_{\mathrm{c}}}}$$

$$= 2^{\frac{(p_{\mathrm{c},2}-p_{\mathrm{c},0})}{\mu_{\mathrm{c}}} - \frac{(p_{\mathrm{c},1}-p_{\mathrm{c},0})}{\mu_{\mathrm{c}}}}$$

$$= 2^{(p_{\mathrm{c},2}-p_{\mathrm{c},1})/\mu_{\mathrm{c}}}$$

Definition 236 (Chromatic pitch interval) *If* $p_{\mathrm{c},1}$ *and* $p_{\mathrm{c},2}$ *are two chromatic pitches in a well-formed pitch system* ψ*, then the* chromatic pitch interval from $p_{\mathrm{c},1}$ to $p_{\mathrm{c},2}$ *is defined and denoted as follows:*

$$\Delta\,\mathrm{p_c}\left(p_{\mathrm{c},1}, p_{\mathrm{c},2}\right) = p_{\mathrm{c},2} - p_{\mathrm{c},1}$$

Theorem 237 *If* Δp_{c} *is a chromatic pitch interval in a pitch system* ψ *then* Δp_{c} *can only take any integer value.*

Proof

R1	Let		$\Delta p_{\mathrm{c}} = \Delta\,\mathrm{p_c}\left(p_{\mathrm{c},1}, p_{\mathrm{c},2}\right)$ where $p_{\mathrm{c},1}$ and $p_{\mathrm{c},2}$ are any two chromatic pitches in ψ.
R2	R1 & 236	$\Rightarrow$	$\Delta\,\mathrm{p_c}\left(p_{\mathrm{c},1}, p_{\mathrm{c},2}\right) = p_{\mathrm{c},2} - p_{\mathrm{c},1}$
R3	62	$\Rightarrow$	$p_{\mathrm{c},1}$ can only take any integer value.
R4	62	$\Rightarrow$	$p_{\mathrm{c},2}$ can only take any integer value.
R5	R2, R3 & R4	$\Rightarrow$	$\Delta\,\mathrm{p_c}\left(p_{\mathrm{c},1}, p_{\mathrm{c},2}\right)$ can only take any integer value.
R6	R5 & R1	$\Rightarrow$	Δp_{c} can only take any integer value.

Intervals between two morphetic pitches

Definition 238 (Definition of $\Delta\,\mathrm{m}\left(p_{\mathrm{m},1}, p_{\mathrm{m},2}\right)$) *If* $p_{\mathrm{m},1}$ *and* $p_{\mathrm{m},2}$ *are two morphetic pitches in a pitch system* ψ *then the* morph interval from $p_{\mathrm{m},1}$ to $p_{\mathrm{m},2}$ *is defined and denoted as follows:*

$$\Delta\,\mathrm{m}\left(p_{\mathrm{m},1}, p_{\mathrm{m},2}\right) = \Delta\,\mathrm{m}\left(\mathrm{m}\left(p_{\mathrm{m},1}\right), \mathrm{m}\left(p_{\mathrm{m},2}\right)\right)$$

Theorem 239 (Formula for $\Delta\,\mathrm{m}\,(p_{\mathrm{m},1}, p_{\mathrm{m},2})$) *If $p_{\mathrm{m},1}$ and $p_{\mathrm{m},2}$ are two morphetic pitches in a pitch system*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

then the morph interval from $p_{\mathrm{m},1}$ to $p_{\mathrm{m},2}$ is given by:

$$\Delta\,\mathrm{m}\,(p_{\mathrm{m},1}, p_{\mathrm{m},2}) = (p_{\mathrm{m},2} - p_{\mathrm{m},1}) \bmod \mu_{\mathrm{m}}$$

Proof

R1	238	$\Rightarrow$	$\Delta\,\mathrm{m}\,(p_{\mathrm{m},1}, p_{\mathrm{m},2}) = \Delta\,\mathrm{m}\,(\mathrm{m}\,(p_{\mathrm{m},1})\,, \mathrm{m}\,(p_{\mathrm{m},2}))$
R2	R1 & 217	$\Rightarrow$	$\Delta\,\mathrm{m}\,(p_{\mathrm{m},1}, p_{\mathrm{m},2}) = (\mathrm{m}\,(p_{\mathrm{m},2}) - \mathrm{m}\,(p_{\mathrm{m},1})) \bmod \mu_{\mathrm{m}}$
R3	97	$\Rightarrow$	$\mathrm{m}\,(p_{\mathrm{m},1}) = p_{\mathrm{m},1} \bmod \mu_{\mathrm{m}}$
R4	97	$\Rightarrow$	$\mathrm{m}\,(p_{\mathrm{m},2}) = p_{\mathrm{m},2} \bmod \mu_{\mathrm{m}}$
R5	R2, R3 & R4	$\Rightarrow$	$\Delta\,\mathrm{m}\,(p_{\mathrm{m},1}, p_{\mathrm{m},2}) = (p_{\mathrm{m},2} \bmod \mu_{\mathrm{m}} - p_{\mathrm{m},1} \bmod \mu_{\mathrm{m}}) \bmod \mu_{\mathrm{m}}$
R6	R5 & 38	$\Rightarrow$	$\Delta\,\mathrm{m}\,(p_{\mathrm{m},1}, p_{\mathrm{m},2}) = (p_{\mathrm{m},2} - p_{\mathrm{m},1} \bmod \mu_{\mathrm{m}}) \bmod \mu_{\mathrm{m}}$
R7	R6 & 38	$\Rightarrow$	$\Delta\,\mathrm{m}\,(p_{\mathrm{m},1}, p_{\mathrm{m},2}) = (p_{\mathrm{m},2} - p_{\mathrm{m},1}) \bmod \mu_{\mathrm{m}}$

Definition 240 (Morphetic pitch interval) *If $p_{\mathrm{m},1}$ and $p_{\mathrm{m},2}$ are two morphetic pitches in a well-formed pitch system ψ, then the* morphetic pitch interval from $p_{\mathrm{m},1}$ to $p_{\mathrm{m},2}$ *is defined and denoted as follows:*

$$\Delta\,\mathrm{p_m}\,(p_{\mathrm{m},1}, p_{\mathrm{m},2}) = p_{\mathrm{m},2} - p_{\mathrm{m},1}$$

Theorem 241 *If Δp_{m} is a morphetic pitch interval in a pitch system ψ then Δp_{m} can only take any integer value.*

Proof

R1	Let		$\Delta p_{\mathrm{m}} = \Delta\,\mathrm{p_m}\,(p_{\mathrm{m},1}, p_{\mathrm{m},2})$ where $p_{\mathrm{m},1}$ and $p_{\mathrm{m},2}$ are any two morphetic pitches in ψ.
R2	R1 & 240	$\Rightarrow$	$\Delta\,\mathrm{p_m}\,(p_{\mathrm{m},1}, p_{\mathrm{m},2}) = p_{\mathrm{m},2} - p_{\mathrm{m},1}$
R3	62	$\Rightarrow$	$p_{\mathrm{m},1}$ can only take any integer value.
R4	62	$\Rightarrow$	$p_{\mathrm{m},2}$ can only take any integer value.
R5	R2, R3 & R4	$\Rightarrow$	$\Delta\,\mathrm{p_m}\,(p_{\mathrm{m},1}, p_{\mathrm{m},2})$ can only take any integer value.
R6	R5 & R1	$\Rightarrow$	Δp_{m} can only take any integer value.

Intervals between two frequencies

Definition 242 ($\Delta \mathrm{f}(f_1, f_2)$) *If f_1 and f_2 are two frequencies within a pitch system ψ then the* frequency interval from f_1 to f_2 *is defined and denoted as follows:*

$$\Delta \mathrm{f}(f_1, f_2) = \frac{f_2}{f_1}$$

Theorem 243 *If f_1 and f_2 are any two frequencies in a pitch system ψ and*

$$\Delta f = \Delta \mathrm{f}(f_1, f_2)$$

then Δf can only take any real value greater than zero.
Proof

R1	Let		$\Delta f = \Delta \mathrm{f}(f_1, f_2)$ where f_1 and f_2 are any two frequencies in ψ.
R2	R1 & 242	$\Rightarrow$	$\Delta f = \frac{f_2}{f_1}$
R3	67	$\Rightarrow$	f_1 and f_2 can only take any real values greater than zero.
R4	R2 & R3	$\Rightarrow$	Δf can only take any real value greater than zero.

Definition 244 (Definition of $\Delta \mathrm{p_c}(f_1, f_2)$) *If f_1 and f_2 are two frequencies within a pitch system ψ then the* chromatic pitch interval from f_1 to f_2 *is defined and denoted as follows:*

$$\Delta \mathrm{p_c}(f_1, f_2) = \Delta \mathrm{p_c}(\mathrm{p_c}(f_1), \mathrm{p_c}(f_2))$$

Theorem 245 (Formula for $\Delta \mathrm{p_c}(f_1, f_2)$) *If f_1 and f_2 are two frequencies within a pitch system*

$$\psi = [\mu_\mathrm{c}, \mu_\mathrm{m}, f_0, p_{\mathrm{c},0}]$$

then the chromatic pitch interval from f_1 to f_2 can be calculated using the following formula:

$$\Delta \mathrm{p_c}(f_1, f_2) = \mu_\mathrm{c} \times \frac{\ln(f_2/f_1)}{\ln 2}$$

Proof

R1	244	$\Rightarrow$	$\Delta \mathrm{p_c}(f_1, f_2) = \Delta \mathrm{p_c}(\mathrm{p_c}(f_1), \mathrm{p_c}(f_2))$
R2	99	$\Rightarrow$	$\mathrm{p_c}(f_1) = \mu_\mathrm{c} \times \frac{\ln(f_1/f_0)}{\ln 2} + p_{\mathrm{c},0}$
R3	99	$\Rightarrow$	$\mathrm{p_c}(f_2) = \mu_\mathrm{c} \times \frac{\ln(f_2/f_0)}{\ln 2} + p_{\mathrm{c},0}$
R4	236	$\Rightarrow$	$\Delta \mathrm{p_c}(\mathrm{p_c}(f_1), \mathrm{p_c}(f_2)) = \mathrm{p_c}(f_2) - \mathrm{p_c}(f_1)$
R5	R2, R3 & R4	$\Rightarrow$	$\Delta \mathrm{p_c}(\mathrm{p_c}(f_1), \mathrm{p_c}(f_2)) = \mu_\mathrm{c} \times \frac{\ln(f_2/f_0)}{\ln 2} + p_{\mathrm{c},0} - \left(\mu_\mathrm{c} \times \frac{\ln(f_1/f_0)}{\ln 2} + p_{\mathrm{c},0}\right)$
			$= \frac{\mu_\mathrm{c}}{\ln 2} \times (\ln(f_2/f_0) - \ln(f_1/f_0))$
			$= \frac{\mu_\mathrm{c}}{\ln 2} \times \ln\left(\frac{f_2}{f_0} \times \frac{f_0}{f_1}\right) = \mu_\mathrm{c} \times \frac{\ln(f_2/f_1)}{\ln 2}$

Definition 246 (Definition of $\Delta\,\mathrm{c}(f_1, f_2)$) *If f_1 and f_2 are two frequencies within a pitch system ψ then the* chroma interval from f_1 to f_2 *is defined and denoted as follows:*

$$\Delta\,\mathrm{c}(f_1, f_2) = \Delta\,\mathrm{c}(\mathrm{c}(f_1), \mathrm{c}(f_2))$$

Theorem 247 (Formula for $\Delta\,\mathrm{c}(f_1, f_2)$) *If f_1 and f_2 are two frequencies within a pitch system*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

then the chroma interval from f_1 to f_2 is given by the following formula:

$$\Delta\,\mathrm{c}(f_1, f_2) = \left(\mu_{\mathrm{c}} \times \frac{\ln(f_2/f_1)}{\ln 2}\right) \bmod \mu_{\mathrm{c}}$$

Proof

R1	246	$\Rightarrow$	$\Delta\,\mathrm{c}(f_1, f_2) = \Delta\,\mathrm{c}(\mathrm{c}(f_1), \mathrm{c}(f_2))$
R2	104	$\Rightarrow$	$\mathrm{c}(f_1) = \mathrm{p_c}(f_1) \bmod \mu_{\mathrm{c}}$
R3	104	$\Rightarrow$	$\mathrm{c}(f_2) = \mathrm{p_c}(f_2) \bmod \mu_{\mathrm{c}}$
R4	213	$\Rightarrow$	$\Delta\,\mathrm{c}(\mathrm{c}(f_1), \mathrm{c}(f_2)) = (\mathrm{c}(f_2) - \mathrm{c}(f_1)) \bmod \mu_{\mathrm{c}}$
R5	R2, R3 & R4	$\Rightarrow$	$\Delta\,\mathrm{c}(\mathrm{c}(f_1), \mathrm{c}(f_2)) = (\mathrm{p_c}(f_2) \bmod \mu_{\mathrm{c}} - \mathrm{p_c}(f_1) \bmod \mu_{\mathrm{c}}) \bmod \mu_{\mathrm{c}}$
R6	R5 & 38	$\Rightarrow$	$\Delta\,\mathrm{c}(\mathrm{c}(f_1), \mathrm{c}(f_2)) = (\mathrm{p_c}(f_2) - \mathrm{p_c}(f_1) \bmod \mu_{\mathrm{c}}) \bmod \mu_{\mathrm{c}}$
R7	R6 & 38	$\Rightarrow$	$\Delta\,\mathrm{c}(\mathrm{c}(f_1), \mathrm{c}(f_2)) = (\mathrm{p_c}(f_2) - \mathrm{p_c}(f_1)) \bmod \mu_{\mathrm{c}}$
R8	236 & 98	$\Rightarrow$	$\mathrm{p_c}(f_2) - \mathrm{p_c}(f_1) = \Delta\,\mathrm{p_c}(\mathrm{p_c}(f_1), \mathrm{p_c}(f_2))$
R9	244	$\Rightarrow$	$\Delta\,\mathrm{p_c}(f_1, f_2) = \Delta\,\mathrm{p_c}(\mathrm{p_c}(f_1), \mathrm{p_c}(f_2))$
R10	245	$\Rightarrow$	$\Delta\,\mathrm{p_c}(f_1, f_2) = \mu_{\mathrm{c}} \times \frac{\ln(f_2/f_1)}{\ln 2}$
R11	R10, R9, R8, R7 & R1	$\Rightarrow$	$\Delta\,\mathrm{c}(f_1, f_2) = \left(\mu_{\mathrm{c}} \times \frac{\ln(f_2/f_1)}{\ln 2}\right) \bmod \mu_{\mathrm{c}}$

Theorem 248 (Second formula for $\Delta\,\mathrm{c}(f_1, f_2)$) *If f_1 and f_2 are two frequencies within a pitch system*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

then the chroma interval from f_1 to f_2 is given by the following formula:

$$\Delta\,\mathrm{c}(f_1, f_2) = \mu_{\mathrm{c}} \times \left(\frac{\ln(f_2/f_1)}{\ln 2} - \mathrm{int}\left(\frac{\ln(f_2/f_1)}{\ln 2}\right)\right)$$

Proof

R1	247	$\Rightarrow$	$\Delta\,\mathrm{c}\,(f_1, f_2) = \left(\mu_\mathrm{c} \times \frac{\ln(f_2/f_1)}{\ln 2}\right) \bmod \mu_\mathrm{c}$
R2	R1 & 33	$\Rightarrow$	$\Delta\,\mathrm{c}\,(f_1, f_2) = \left(\mu_\mathrm{c} \times \frac{\ln(f_2/f_1)}{\ln 2}\right) - \mu_\mathrm{c} \times \mathrm{int}\left(\frac{\mu_\mathrm{c} \times \ln(f_2/f_1)}{\mu_\mathrm{c} \times \ln 2}\right)$
			$= \mu_\mathrm{c} \times \left(\frac{\ln(f_2/f_1)}{\ln 2} - \mathrm{int}\left(\frac{\ln(f_2/f_1)}{\ln 2}\right)\right)$

Intervals between two pitches

Definition 249 (Definition of $\Delta\,\mathrm{c}\,(p_1, p_2)$) *If p_1 and p_2 are two pitches in a pitch system ψ then the* chroma interval from p_1 to p_2 *is defined and denoted as follows:*

$$\Delta\,\mathrm{c}\,(p_1, p_2) = \Delta\,\mathrm{c}\,(\mathrm{c}\,(p_1)\,, \mathrm{c}\,(p_2))$$

Theorem 250 (Formula for $\Delta\,\mathrm{c}\,(p_1, p_2)$) *If p_1 and p_2 are two pitches in a pitch system $\psi = [\mu_\mathrm{c}, \mu_\mathrm{m}, f_0, p_{\mathrm{c},0}]$ then the chroma interval from p_1 to p_2 is given by the following expression:*

$$\Delta\,\mathrm{c}\,(p_1, p_2) = (\mathrm{p_c}\,(p_2) - \mathrm{p_c}\,(p_1)) \bmod \mu_\mathrm{c}$$

Proof

R1	249	$\Rightarrow$	$\Delta\,\mathrm{c}\,(p_1, p_2) = \Delta\,\mathrm{c}\,(\mathrm{c}\,(p_1)\,, \mathrm{c}\,(p_2))$
R2	R1 & 213	$\Rightarrow$	$\Delta\,\mathrm{c}\,(p_1, p_2) = (\mathrm{c}\,(p_2) - \mathrm{c}\,(p_1)) \bmod \mu_\mathrm{c}$
R3	R2 & 71	$\Rightarrow$	$\Delta\,\mathrm{c}\,(p_1, p_2) = (\mathrm{p_c}\,(p_2) \bmod \mu_\mathrm{c} - \mathrm{p_c}\,(p_1) \bmod \mu_\mathrm{c}) \bmod \mu_\mathrm{c}$
R4	R3 & 38	$\Rightarrow$	$\Delta\,\mathrm{c}\,(p_1, p_2) = (\mathrm{p_c}\,(p_2) - \mathrm{p_c}\,(p_1)) \bmod \mu_\mathrm{c}$

Definition 251 (Definition of $\Delta\,\mathrm{m}\,(p_1, p_2)$) *If p_1 and p_2 are two pitches in a pitch system ψ then the* morph interval from p_1 to p_2 *is defined and denoted as follows:*

$$\Delta\,\mathrm{m}\,(p_1, p_2) = \Delta\,\mathrm{m}\,(\mathrm{m}\,(p_1)\,, \mathrm{m}\,(p_2))$$

Theorem 252 (Formula for $\Delta\,\mathrm{m}\,(p_1, p_2)$) *If p_1 and p_2 are two pitches in a pitch system $\psi = [\mu_\mathrm{c}, \mu_\mathrm{m}, f_0, p_{\mathrm{c},0}]$ then the morph interval from p_1 to p_2 is given by the following expression:*

$$\Delta\,\mathrm{m}\,(p_1, p_2) = (\mathrm{p_m}\,(p_2) - \mathrm{p_m}\,(p_1)) \bmod \mu_\mathrm{m}$$

Proof

R1	251	$\Rightarrow$	$\Delta\,\mathrm{m}\,(p_1, p_2) = \Delta\,\mathrm{m}\,(\mathrm{m}\,(p_1)\,, \mathrm{m}\,(p_2))$
R2	R1 & 217	$\Rightarrow$	$\Delta\,\mathrm{m}\,(p_1, p_2) = (\mathrm{m}\,(p_2) - \mathrm{m}\,(p_1)) \bmod \mu_\mathrm{m}$
R3	R2 & 76	$\Rightarrow$	$\Delta\,\mathrm{m}\,(p_1, p_2) = (\mathrm{p_m}\,(p_2) \bmod \mu_\mathrm{m} - \mathrm{p_m}\,(p_1) \bmod \mu_\mathrm{m}) \bmod \mu_\mathrm{m}$
R4	R3 & 38	$\Rightarrow$	$\Delta\,\mathrm{m}\,(p_1, p_2) = (\mathrm{p_m}\,(p_2) - \mathrm{p_m}\,(p_1)) \bmod \mu_\mathrm{m}$

Definition 253 *If p_1 and p_2 are two pitches in a pitch system ψ then the* chromamorph interval from p_1 to p_2 *is defined and denoted as follows:*

$$\Delta\,\mathrm{q}\,(p_1, p_2) = \Delta\,\mathrm{q}\,(\mathrm{q}\,(p_1)\,, \mathrm{q}\,(p_2))$$

Definition 254 *If p_1 and p_2 are two pitches in a pitch system ψ then the* chromatic genus interval from p_1 to p_2 *is defined and denoted as follows:*

$$\Delta\,\mathrm{g_c}\,(p_1, p_2) = \Delta\,\mathrm{g_c}\,(\mathrm{g}\,(p_1)\,, \mathrm{g}\,(p_2))$$

Theorem 255 *If p_1 and p_2 are two pitches in a pitch system*

$$\psi = [\mu_\mathrm{c}, \mu_\mathrm{m}, f_0, p_{\mathrm{c},0}]$$

then the chromatic genus interval from p_1 to p_2 is given by the following expression:

$$\Delta\,\mathrm{g_c}\,(p_1, p_2) = \mathrm{g_c}\,(p_2) - \mathrm{g_c}\,(p_1) - \mu_\mathrm{c} \times ((\mathrm{m}\,(p_2) - \mathrm{m}\,(p_1))\ \mathrm{div}\ \mu_\mathrm{m})$$

Proof

R1	254	$\Rightarrow$	$\Delta\,\mathrm{g_c}\,(p_1, p_2) = \Delta\,\mathrm{g_c}\,(\mathrm{g}\,(p_1)\,, \mathrm{g}\,(p_2))$
R2	230 & R1	$\Rightarrow$	$\Delta\,\mathrm{g_c}\,(p_1, p_2) = \mathrm{g_c}\,(\mathrm{g}\,(p_2)) - \mathrm{g_c}\,(\mathrm{g}\,(p_1)) - \mu_\mathrm{c} \times ((\mathrm{m}\,(\mathrm{g}\,(p_2)) - \mathrm{m}\,(\mathrm{g}\,(p_1)))\ \mathrm{div}\ \mu_\mathrm{m})$
R3	114 & R2	$\Rightarrow$	$\Delta\,\mathrm{g_c}\,(p_1, p_2) = \mathrm{g_c}\,(p_2) - \mathrm{g_c}\,(p_1) - \mu_\mathrm{c} \times ((\mathrm{m}\,(\mathrm{g}\,(p_2)) - \mathrm{m}\,(\mathrm{g}\,(p_1)))\ \mathrm{div}\ \mu_\mathrm{m})$
R4	116 & R3	$\Rightarrow$	$\Delta\,\mathrm{g_c}\,(p_1, p_2) = \mathrm{g_c}\,(p_2) - \mathrm{g_c}\,(p_1) - \mu_\mathrm{c} \times ((\mathrm{m}\,(p_2) - \mathrm{m}\,(p_1))\ \mathrm{div}\ \mu_\mathrm{m})$

Theorem 256 *If $\Delta g_\mathrm{c} = \Delta\,\mathrm{g_c}\,(p_1, p_2)$ where p_1 and p_2 are any two pitches in*

$$\psi = [\mu_\mathrm{c}, \mu_\mathrm{m}, f_0, p_{\mathrm{c},0}]$$

then Δg_c can only take any integer value.
Proof

R1	Let		$\Delta g_\mathrm{c} = \Delta\,\mathrm{g_c}\,(p_1, p_2)$ where p_1 and p_2 are any two pitches in a pitch system $\psi = [\mu_\mathrm{c}, \mu_\mathrm{m}, f_0, p_{\mathrm{c},0}]$.
R2	R1 & 255	$\Rightarrow$	$\Delta g_\mathrm{c} = \mathrm{g_c}\,(p_2) - \mathrm{g_c}\,(p_1) - \mu_\mathrm{c} \times ((\mathrm{m}\,(p_2) - \mathrm{m}\,(p_1))\ \mathrm{div}\ \mu_\mathrm{m})$
R3	61	$\Rightarrow$	μ_c can only take any positive integer value.
R4	61	$\Rightarrow$	μ_m can only take any positive integer value.
R5	77	$\Rightarrow$	$\mathrm{m}\,(p_1)$ and $\mathrm{m}\,(p_2)$ can each only take any value such that $(0 \leq \mathrm{m}\,(p_1)\,, \mathrm{m}\,(p_2) < \mu_\mathrm{m}) \wedge (\mathrm{m}\,(p_1)\,, \mathrm{m}\,(p_2) \in \mathbb{Z})$.
R6	83	$\Rightarrow$	$\mathrm{g_c}\,(p_2)$ and $\mathrm{g_c}\,(p_1)$ can each only take any integer value.
R7	R2, 48, R3, R4, R5 & R6	$\Rightarrow$	Δg_c can only take any integer value.

Definition 257 (Definition of $\Delta\,\mathrm{g}\,(p_1, p_2)$) *If p_1 and p_2 are two pitches in a pitch system ψ then the* genus interval from p_1 to p_2 *is defined and denoted as follows:*

$$\Delta\,\mathrm{g}\,(p_1, p_2) = \Delta\,\mathrm{g}\,(\mathrm{g}\,(p_1)\,, \mathrm{g}\,(p_2))$$

Theorem 258 (Formula for $\Delta\,\mathrm{g}\,(p_1, p_2)$) *If p_1 and p_2 are two pitches in a pitch system ψ then the genus interval from p_1 to p_2 is given by the following expression:*

$$\Delta\,\mathrm{g}\,(p_1, p_2) = [\Delta\,\mathrm{g_c}\,(p_1, p_2)\,, \Delta\,\mathrm{m}\,(p_1, p_2)]$$

Proof

R1	257	$\Rightarrow$	$\Delta\,\mathrm{g}\,(p_1, p_2) = \Delta\,\mathrm{g}\,(\mathrm{g}\,(p_1)\,, \mathrm{g}\,(p_2))$
R2	R1 & 231	$\Rightarrow$	$\Delta\,\mathrm{g}\,(p_1, p_2) = [\Delta\,\mathrm{g_c}\,(\mathrm{g}\,(p_1)\,, \mathrm{g}\,(p_2))\,, \Delta\,\mathrm{m}\,(\mathrm{g}\,(p_1)\,, \mathrm{g}\,(p_2))]$
R3	R2 & 254	$\Rightarrow$	$\Delta\,\mathrm{g}\,(p_1, p_2) = [\Delta\,\mathrm{g_c}\,(p_1, p_2)\,, \Delta\,\mathrm{m}\,(\mathrm{g}\,(p_1)\,, \mathrm{g}\,(p_2))]$
R4	R3 & 228	$\Rightarrow$	$\Delta\,\mathrm{g}\,(p_1, p_2) = [\Delta\,\mathrm{g_c}\,(p_1, p_2)\,, \Delta\,\mathrm{m}\,(\mathrm{m}\,(\mathrm{g}\,(p_1))\,, \mathrm{m}\,(\mathrm{g}\,(p_2)))]$
R5	R4 & 116	$\Rightarrow$	$\Delta\,\mathrm{g}\,(p_1, p_2) = [\Delta\,\mathrm{g_c}\,(p_1, p_2)\,, \Delta\,\mathrm{m}\,(\mathrm{m}\,(p_1)\,, \mathrm{m}\,(p_2))]$
R6	R5 & 251	$\Rightarrow$	$\Delta\,\mathrm{g}\,(p_1, p_2) = [\Delta\,\mathrm{g_c}\,(p_1, p_2)\,, \Delta\,\mathrm{m}\,(p_1, p_2)]$

Definition 259 (Definition of $\Delta\,\mathrm{p_c}\,(p_1, p_2)$) *If p_1 and p_2 are two pitches in a pitch system ψ then the* chromatic pitch interval from p_1 to p_2 *is defined and denoted as follows:*

$$\Delta\,\mathrm{p_c}\,(p_1, p_2) = \Delta\,\mathrm{p_c}\,(\mathrm{p_c}\,(p_1)\,, \mathrm{p_c}\,(p_2))$$

Theorem 260 (Formula for $\Delta\,\mathrm{p_c}\,(p_1, p_2)$) *If p_1 and p_2 are two pitches in a pitch system ψ then the chromatic pitch interval from p_1 to p_2 is given by*

$$\Delta\,\mathrm{p_c}\,(p_1, p_2) = \mathrm{p_c}\,(p_2) - \mathrm{p_c}\,(p_1)$$

Proof

R1	259	$\Rightarrow$	$\Delta\,\mathrm{p_c}\,(p_1, p_2) = \Delta\,\mathrm{p_c}\,(\mathrm{p_c}\,(p_1)\,, \mathrm{p_c}\,(p_2))$
R2	R1 & 236	$\Rightarrow$	$\Delta\,\mathrm{p_c}\,(p_1, p_2) = \mathrm{p_c}\,(p_2) - \mathrm{p_c}\,(p_1)$

Definition 261 (Definition of $\Delta\,\mathrm{p_m}\,(p_1, p_2)$) *If p_1 and p_2 are two pitches in a pitch system ψ then the* morphetic pitch interval from p_1 to p_2 *is defined and denoted as follows:*

$$\Delta\,\mathrm{p_m}\,(p_1, p_2) = \Delta\,\mathrm{p_m}\,(\mathrm{p_m}\,(p_1)\,, \mathrm{p_m}\,(p_2))$$

Theorem 262 (Formula for $\Delta\,\mathrm{p_m}\,(p_1, p_2)$) *If p_1 and p_2 are two pitches in a pitch system ψ then the morphetic pitch interval from p_1 to p_2 is given by*

$$\Delta\,\mathrm{p_m}\,(p_1, p_2) = \mathrm{p_m}\,(p_2) - \mathrm{p_m}\,(p_1)$$

Proof

R1	261	$\Rightarrow$	$\Delta\,\mathrm{p_m}(p_1,p_2)=\Delta\,\mathrm{p_m}(\mathrm{p_m}(p_1),\mathrm{p_m}(p_2))$
R2	R1 & 240	$\Rightarrow$	$\Delta\,\mathrm{p_m}(p_1,p_2)=\mathrm{p_m}(p_2)-\mathrm{p_m}(p_1)$

Definition 263 (Definition of $\Delta\,\mathrm{f}(p_1,p_2)$) *If p_1 and p_2 are two pitches in a pitch system ψ then the* frequency interval from p_1 to p_2 *is defined and denoted as follows:*

$$\Delta\,\mathrm{f}(p_1,p_2)=\Delta\,\mathrm{f}(\mathrm{f}(p_1),\mathrm{f}(p_2))$$

Theorem 264 (Formula for $\Delta\,\mathrm{f}(p_1,p_2)$) *If p_1 and p_2 are two pitches in a pitch system*

$$\psi=[\mu_\mathrm{c},\mu_\mathrm{m},f_0,p_{\mathrm{c},0}]$$

then the frequency interval from p_1 to p_2 is given by the following formula:

$$\Delta\,\mathrm{f}(p_1,p_2)=2^{(\mathrm{p_c}(p_2)-\mathrm{p_c}(p_1))/\mu_\mathrm{c}}$$

Proof

R1	263	$\Rightarrow$	$\Delta\,\mathrm{f}(p_1,p_2)=\Delta\,\mathrm{f}(\mathrm{f}(p_1),\mathrm{f}(p_2))$
R2	R1 & 242	$\Rightarrow$	$\Delta\,\mathrm{f}(p_1,p_2)=\frac{\mathrm{f}(p_1)}{\mathrm{f}(p_2)}$
R3	R2 & 66	$\Rightarrow$	$\Delta\,\mathrm{f}(p_1,p_2)=\frac{f_0\times 2^{(\mathrm{p_c}(p_2)-p_{\mathrm{c},0})/\mu_\mathrm{c}}}{f_0\times 2^{(\mathrm{p_c}(p_1)-p_{\mathrm{c},0})/\mu_\mathrm{c}}}$

$$=\frac{2^{(\mathrm{p_c}(p_2)-p_{\mathrm{c},0})/\mu_\mathrm{c}}}{2^{(\mathrm{p_c}(p_1)-p_{\mathrm{c},0})/\mu_\mathrm{c}}}$$

$$=2^{\frac{\mathrm{p_c}(p_2)-p_{\mathrm{c},0}}{\mu_\mathrm{c}}-\frac{\mathrm{p_c}(p_1)-p_{\mathrm{c},0}}{\mu_\mathrm{c}}}$$

$$=2^{(\mathrm{p_c}(p_2)-\mathrm{p_c}(p_1))/\mu_\mathrm{c}}$$

Definition 265 (Pitch interval) *If p_1 and p_2 are two pitches in a pitch system ψ then the* pitch interval from p_1 to p_2 *is defined and denoted as follows:*

$$\Delta\,\mathrm{p}(p_1,p_2)=[\Delta\,\mathrm{p_c}(p_1,p_2),\Delta\,\mathrm{p_m}(p_1,p_2)]$$

4.4.2 Derived *MIPS* intervals

Deriving *MIPS* intervals from a pitch interval

Definition 266 (Chromatic pitch interval of a pitch interval) *If p_1 and p_2 are any two pitches in a pitch system ψ then*

$$\Delta p=\Delta\,\mathrm{p}(p_1,p_2)\Rightarrow\Delta\,\mathrm{p_c}(\Delta p)=\Delta\,\mathrm{p_c}(p_1,p_2)$$

Theorem 267 (Formula for $\Delta\,\mathrm{p_c}(\Delta p)$) *If $\Delta p=[\Delta p_\mathrm{c},\Delta p_\mathrm{m}]$ in a pitch system ψ then*

$$\Delta\,\mathrm{p_c}(\Delta p)=\Delta p_\mathrm{c}$$

Proof

R1	Let		$\Delta p = \Delta\,\mathrm{p}\,(p_1, p_2)$
R2	Let		$\Delta p = [\Delta p_{\mathrm{c}}, \Delta p_{\mathrm{m}}]$
R3	R1 & 266	$\Rightarrow$	$\Delta\,\mathrm{p_c}\,(\Delta p) = \Delta\,\mathrm{p_c}\,(p_1, p_2)$
R4	259	$\Rightarrow$	$\Delta\,\mathrm{p_c}\,(p_1, p_2) = \Delta\,\mathrm{p_c}\,(\mathrm{p_c}\,(p_1)\,, \mathrm{p_c}\,(p_2))$
R5	265	$\Rightarrow$	$\Delta\,\mathrm{p}\,(p_1, p_2) = [\Delta\,\mathrm{p_c}\,(p_1, p_2)\,, \Delta\,\mathrm{p_m}\,(p_1, p_2)]$
R6	R4 & 261 & R5	$\Rightarrow$	$\Delta\,\mathrm{p}\,(p_1, p_2) = [\Delta\,\mathrm{p_c}\,(\mathrm{p_c}\,(p_1)\,, \mathrm{p_c}\,(p_2))\,, \Delta\,\mathrm{p_m}\,(\mathrm{p_m}\,(p_1)\,, \mathrm{p_m}\,(p_2))]$
R7	R1 & R2	$\Rightarrow$	$\Delta\,\mathrm{p}\,(p_1, p_2) = [\Delta p_{\mathrm{c}}, \Delta p_{\mathrm{m}}]$
R8	R6 & R7	$\Rightarrow$	$\Delta\,\mathrm{p_c}\,(\mathrm{p_c}\,(p_1)\,, \mathrm{p_c}\,(p_2)) = \Delta p_{\mathrm{c}}$
R9	R8 & R4	$\Rightarrow$	$\Delta\,\mathrm{p_c}\,(p_1, p_2) = \Delta p_{\mathrm{c}}$
R10	R9 & R3	$\Rightarrow$	$\Delta\,\mathrm{p_c}\,(\Delta p) = \Delta p_{\mathrm{c}}$

Definition 268 (Morphetic pitch interval of a pitch interval) *If p_1 and p_2 are any two pitches in a pitch system ψ then*

$$\Delta p = \Delta\,\mathrm{p}\,(p_1, p_2) \Rightarrow \Delta\,\mathrm{p_m}\,(\Delta p) = \Delta\,\mathrm{p_m}\,(p_1, p_2)$$

Theorem 269 (Formula for $\Delta\,\mathrm{p_m}\,(\Delta p)$) *If $\Delta p = [\Delta p_{\mathrm{c}}, \Delta p_{\mathrm{m}}]$ in a pitch system ψ then*

$$\Delta\,\mathrm{p_m}\,(\Delta p) = \Delta p_{\mathrm{m}}$$

Proof

R1	Let		$\Delta p = \Delta\,\mathrm{p}\,(p_1, p_2)$
R2	Let		$\Delta p = [\Delta p_\mathrm{c}, \Delta p_\mathrm{m}]$
R3	R1 & 268	$\Rightarrow$	$\Delta\,\mathrm{p_m}\,(\Delta p) = \Delta\,\mathrm{p_m}\,(p_1, p_2)$
R4	261	$\Rightarrow$	$\Delta\,\mathrm{p_m}\,(p_1, p_2) = \Delta\,\mathrm{p_m}\,(\mathrm{p_m}\,(p_1)\,, \mathrm{p_m}\,(p_2))$
R5	265	$\Rightarrow$	$\Delta\,\mathrm{p}\,(p_1, p_2) = [\Delta\,\mathrm{p_c}\,(p_1, p_2)\,, \Delta\,\mathrm{p_m}\,(p_1, p_2)]$
R6	R4 & 261 & R5	$\Rightarrow$	$\Delta\,\mathrm{p}\,(p_1, p_2) = [\Delta\,\mathrm{p_c}\,(\mathrm{p_c}\,(p_1)\,, \mathrm{p_c}\,(p_2))\,, \Delta\,\mathrm{p_m}\,(\mathrm{p_m}\,(p_1)\,, \mathrm{p_m}\,(p_2))]$
R7	R1 & R2	$\Rightarrow$	$\Delta\,\mathrm{p}\,(p_1, p_2) = [\Delta p_\mathrm{c}, \Delta p_\mathrm{m}]$
R8	R6 & R7	$\Rightarrow$	$\Delta\,\mathrm{p_m}\,(\mathrm{p_m}\,(p_1)\,, \mathrm{p_m}\,(p_2)) = \Delta p_\mathrm{m}$
R9	R8 & R4	$\Rightarrow$	$\Delta\,\mathrm{p_m}\,(p_1, p_2) = \Delta p_\mathrm{m}$
R10	R9 & R3	$\Rightarrow$	$\Delta\,\mathrm{p_m}\,(\Delta p) = \Delta p_\mathrm{m}$

Theorem 270 *If ψ is a pitch system and Δp is a pitch interval in ψ then*

$$\Delta p = [\Delta\,\mathrm{p_c}\,(\Delta p)\,, \Delta\,\mathrm{p_m}\,(\Delta p)]$$

Proof

R1	Let		$\Delta p = [\Delta p_\mathrm{c}, \Delta p_\mathrm{m}]$
R2	R1 & 267	$\Rightarrow$	$\Delta\,\mathrm{p_c}\,(\Delta p) = \Delta p_\mathrm{c}$
R3	R1 & 269	$\Rightarrow$	$\Delta\,\mathrm{p_m}\,(\Delta p) = \Delta p_\mathrm{m}$
R4	R1, R2 & R3	$\Rightarrow$	$\Delta p = [\Delta\,\mathrm{p_c}\,(\Delta p)\,, \Delta\,\mathrm{p_m}\,(\Delta p)]$

Definition 271 (Definition of $\Delta\,\mathrm{f}\,(\Delta p)$) *If p_1 and p_2 are any two pitches in a pitch system ψ then*

$$\Delta p = \Delta\,\mathrm{p}\,(p_1, p_2) \Rightarrow \Delta\,\mathrm{f}\,(\Delta p) = \Delta\,\mathrm{f}\,(p_1, p_2)$$

Theorem 272 (Formula for $\Delta\,\mathrm{f}\,(\Delta p)$) *If Δp is a pitch interval in a pitch system ψ then*

$$\Delta\,\mathrm{f}\,(\Delta p) = 2^{\Delta\,\mathrm{p_c}(\Delta p)/\mu_\mathrm{c}}$$

Proof

R1	Let		$\Delta p = \Delta\,\mathrm{p}\,(p_1, p_2)$
R2	R1 & 271	$\Rightarrow$	$\Delta\,\mathrm{f}\,(\Delta p) = \Delta\,\mathrm{f}\,(p_1, p_2)$
R3	264	$\Rightarrow$	$\Delta\,\mathrm{f}\,(p_1, p_2) = 2^{(\mathrm{p_c}(p_2) - \mathrm{p_c}(p_1))/\mu_\mathrm{c}}$
R4	R1 & 266	$\Rightarrow$	$\Delta\,\mathrm{p_c}\,(\Delta p) = \Delta\,\mathrm{p_c}\,(p_1, p_2)$
R5	260	$\Rightarrow$	$\Delta\,\mathrm{p_c}\,(p_1, p_2) = \mathrm{p_c}\,(p_2) - \mathrm{p_c}\,(p_1)$
R6	R5 & R4	$\Rightarrow$	$\Delta\,\mathrm{p_c}\,(\Delta p) = \mathrm{p_c}\,(p_2) - \mathrm{p_c}\,(p_1)$
R7	R6 & R3	$\Rightarrow$	$\Delta\,\mathrm{f}\,(p_1, p_2) = 2^{\Delta\,\mathrm{p_c}(\Delta p)/\mu_\mathrm{c}}$
R8	R7 & R2	$\Rightarrow$	$\Delta\,\mathrm{f}\,(\Delta p) = 2^{\Delta\,\mathrm{p_c}(\Delta p)/\mu_\mathrm{c}}$

Definition 273 (Definition of $\Delta\,\mathrm{c}\,(\Delta p)$) *If p_1 and p_2 are any two pitches in a pitch system ψ then*

$$\Delta p = \Delta\,\mathrm{p}\,(p_1, p_2) \Rightarrow \Delta\,\mathrm{c}\,(\Delta p) = \Delta\,\mathrm{c}\,(p_1, p_2)$$

Theorem 274 (Formula for $\Delta\,\mathrm{c}\,(\Delta p)$) *If Δp is a pitch interval in a pitch system ψ then*

$$\Delta\,\mathrm{c}\,(\Delta p) = \Delta\,\mathrm{p_c}\,(\Delta p) \bmod \mu_\mathrm{c}$$

Proof

R1	Let		$\Delta p = \Delta\,\mathrm{p}\,(p_1, p_2)$
R2	R1 & 273	$\Rightarrow$	$\Delta\,\mathrm{c}\,(\Delta p) = \Delta\,\mathrm{c}\,(p_1, p_2)$
R3	R2 & 250	$\Rightarrow$	$\Delta\,\mathrm{c}\,(\Delta p) = (\mathrm{p_c}\,(p_2) - \mathrm{p_c}\,(p_1)) \bmod \mu_\mathrm{c}$
R4	R1 & 266	$\Rightarrow$	$\Delta\,\mathrm{p_c}\,(\Delta p) = \Delta\,\mathrm{p_c}\,(p_1, p_2)$
R5	R4 & 260	$\Rightarrow$	$\Delta\,\mathrm{p_c}\,(\Delta p) = \mathrm{p_c}\,(p_2) - \mathrm{p_c}\,(p_1)$
R6	R5 & R3	$\Rightarrow$	$\Delta\,\mathrm{c}\,(\Delta p) = \Delta\,\mathrm{p_c}\,(\Delta p) \bmod \mu_\mathrm{c}$

Definition 275 (Definition of $\Delta\,\mathrm{m}\,(\Delta p)$) *If p_1 and p_2 are any two pitches in a pitch system ψ then*

$$\Delta p = \Delta\,\mathrm{p}\,(p_1, p_2) \Rightarrow \Delta\,\mathrm{m}\,(\Delta p) = \Delta\,\mathrm{m}\,(p_1, p_2)$$

Theorem 276 (Formula for $\Delta\,\mathrm{m}\,(\Delta p)$) *If Δp is a pitch interval in a pitch system ψ then*

$$\Delta\,\mathrm{m}\,(\Delta p) = \Delta p_\mathrm{m} \Delta p \bmod \mu_\mathrm{m}$$

Proof

R1	Let		$\Delta p = \Delta\,\mathrm{p}\,(p_1, p_2)$
R2	R1 & 275	$\Rightarrow$	$\Delta\,\mathrm{m}\,(\Delta p) = \Delta\,\mathrm{m}\,(p_1, p_2)$
R3	R2 & 252	$\Rightarrow$	$\Delta\,\mathrm{m}\,(\Delta p) = (\mathrm{p_m}\,(p_2) - \mathrm{p_m}\,(p_1)) \bmod \mu_{\mathrm{m}}$
R4	R1 & 268	$\Rightarrow$	$\Delta\,\mathrm{p_m}\,(\Delta p) = \Delta\,\mathrm{p_m}\,(p_1, p_2)$
R5	R4 & 262	$\Rightarrow$	$\Delta\,\mathrm{p_m}\,(\Delta p) = \mathrm{p_m}\,(p_2) - \mathrm{p_m}\,(p_1)$
R6	R5 & R3	$\Rightarrow$	$\Delta\,\mathrm{m}\,(\Delta p) = \Delta\,\mathrm{p_m}\,(\Delta p) \bmod \mu_{\mathrm{m}}$

Definition 277 (Definition of $\Delta\,\mathrm{q}\,(\Delta p)$) *If p_1 and p_2 are any two pitches in a pitch system ψ then*

$$\Delta p = \Delta\,\mathrm{p}\,(p_1, p_2) \Rightarrow \Delta\,\mathrm{q}\,(\Delta p) = \Delta\,\mathrm{q}\,(p_1, p_2)$$

Theorem 278 (Formula for $\Delta\,\mathrm{q}\,(\Delta p)$) *If Δp is a pitch interval in a pitch system ψ then*

$$\Delta\,\mathrm{q}\,(\Delta p) = [\Delta\,\mathrm{c}\,(\Delta p)\,, \Delta\,\mathrm{m}\,(\Delta p)]$$

Proof

R1	Let		$\Delta p = \Delta\,\mathrm{p}\,(p_1, p_2)$
R2	R1 & 275	$\Rightarrow$	$\Delta\,\mathrm{m}\,(\Delta p) = \Delta\,\mathrm{m}\,(p_1, p_2)$
R3	R1 & 273	$\Rightarrow$	$\Delta\,\mathrm{c}\,(\Delta p) = \Delta\,\mathrm{c}\,(p_1, p_2)$
R4	R1 & 277	$\Rightarrow$	$\Delta\,\mathrm{q}\,(\Delta p) = \Delta\,\mathrm{q}\,(p_1, p_2)$
R5	R4 & 253	$\Rightarrow$	$\Delta\,\mathrm{q}\,(\Delta p) = \Delta\,\mathrm{q}\,(\mathrm{q}\,(p_1)\,, \mathrm{q}\,(p_2))$
R6	R5 & 223	$\Rightarrow$	$\Delta\,\mathrm{q}\,(\Delta p) = [\Delta\,\mathrm{c}\,(\mathrm{q}\,(p_1)\,, \mathrm{q}\,(p_2))\,, \Delta\,\mathrm{m}\,(\mathrm{q}\,(p_1)\,, \mathrm{q}\,(p_2))]$
R7	221	$\Rightarrow$	$\Delta\,\mathrm{c}\,(\mathrm{q}\,(p_1)\,, \mathrm{q}\,(p_2)) = \Delta\,\mathrm{c}\,(\mathrm{c}\,(\mathrm{q}\,(p_1))\,, \mathrm{c}\,(\mathrm{q}\,(p_2)))$
R8	105 & R7	$\Rightarrow$	$\Delta\,\mathrm{c}\,(\mathrm{q}\,(p_1)\,, \mathrm{q}\,(p_2)) = \Delta\,\mathrm{c}\,(\mathrm{c}\,(p_1)\,, \mathrm{c}\,(p_2))$
R9	249 & R8	$\Rightarrow$	$\Delta\,\mathrm{c}\,(\mathrm{q}\,(p_1)\,, \mathrm{q}\,(p_2)) = \Delta\,\mathrm{c}\,(p_1, p_2)$
R10	R9 & R3	$\Rightarrow$	$\Delta\,\mathrm{c}\,(\mathrm{q}\,(p_1)\,, \mathrm{q}\,(p_2)) = \Delta\,\mathrm{c}\,(\Delta p)$
R11	222	$\Rightarrow$	$\Delta\,\mathrm{m}\,(\mathrm{q}\,(p_1)\,, \mathrm{q}\,(p_2)) = \Delta\,\mathrm{m}\,(\mathrm{m}\,(\mathrm{q}\,(p_1))\,, \mathrm{m}\,(\mathrm{q}\,(p_2)))$
R12	107 & R11	$\Rightarrow$	$\Delta\,\mathrm{m}\,(\mathrm{q}\,(p_1)\,, \mathrm{q}\,(p_2)) = \Delta\,\mathrm{m}\,(\mathrm{m}\,(p_1)\,, \mathrm{m}\,(p_2))$
R13	251 & R12	$\Rightarrow$	$\Delta\,\mathrm{m}\,(\mathrm{q}\,(p_1)\,, \mathrm{q}\,(p_2)) = \Delta\,\mathrm{m}\,(p_1, p_2)$
R14	R13 & R2	$\Rightarrow$	$\Delta\,\mathrm{m}\,(\mathrm{q}\,(p_1)\,, \mathrm{q}\,(p_2)) = \Delta\,\mathrm{m}\,(\Delta p)$
R15	R6, R10 & R14	$\Rightarrow$	$\Delta\,\mathrm{q}\,(\Delta p) = [\Delta\,\mathrm{c}\,(\Delta p)\,, \Delta\,\mathrm{m}\,(\Delta p)]$

Definition 279 (Chromatic genus interval of a pitch interval) *If p_1 and p_2 are any two pitches in a pitch system ψ then*

$$\Delta p = \Delta\,\mathrm{p}\,(p_1, p_2) \Rightarrow \Delta\,\mathrm{g_c}\,(\Delta p) = \Delta\,\mathrm{g_c}\,(p_1, p_2)$$

Theorem 280 (Formula for $\Delta\,\mathrm{g_c}\,(\Delta p)$) *If Δp is a pitch interval in*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

then:

$$\Delta\,\mathrm{g_c}\,(\Delta p) = \Delta\,\mathrm{p_c}\,(\Delta p) - \mu_{\mathrm{c}} \times (\Delta\,\mathrm{p_m}\,(\Delta p) \;\mathrm{div}\; \mu_{\mathrm{m}})$$

Proof

R1	Let		$\Delta p = \Delta\,\mathrm{p}\,(p_1, p_2)$
R2	R1 & 279	$\Rightarrow$	$\Delta\,\mathrm{g_c}\,(\Delta p) = \Delta\,\mathrm{g_c}\,(p_1, p_2)$
R3	R2 & 255	$\Rightarrow$	$\Delta\,\mathrm{g_c}\,(\Delta p) = \mathrm{g_c}\,(p_2) - \mathrm{g_c}\,(p_1) - \mu_\mathrm{c} \times ((\mathrm{m}\,(p_2) - \mathrm{m}\,(p_1)) \mathrm{~div~} \mu_\mathrm{m})$
R4	R1 & 266	$\Rightarrow$	$\Delta\,\mathrm{p_c}\,(\Delta p) = \Delta\,\mathrm{p_c}\,(p_1, p_2)$
R5	R4 & 260	$\Rightarrow$	$\Delta\,\mathrm{p_c}\,(\Delta p) = \mathrm{p_c}\,(p_2) - \mathrm{p_c}\,(p_1)$
R6	R1 & 268	$\Rightarrow$	$\Delta\,\mathrm{p_m}\,(\Delta p) = \Delta\,\mathrm{p_m}\,(p_1, p_2)$
R7	R6 & 262	$\Rightarrow$	$\Delta\,\mathrm{p_m}\,(\Delta p) = \mathrm{p_m}\,(p_2) - \mathrm{p_m}\,(p_1)$
R8	R3 & 82	$\Rightarrow$	$\Delta\,\mathrm{g_c}\,(\Delta p) = \mathrm{p_c}\,(p_2) - \mu_\mathrm{c} \times \mathrm{o_m}\,(p_2) - \mathrm{p_c}\,(p_1) + \mu_\mathrm{c} \times \mathrm{o_m}\,(p_1)$
			$\quad -\mu_\mathrm{c} \times ((\mathrm{m}\,(p_2) - \mathrm{m}\,(p_1)) \mathrm{~div~} \mu_\mathrm{m})$
			$= \mathrm{p_c}\,(p_2) - \mathrm{p_c}\,(p_1) - \mu_\mathrm{c} \times (\mathrm{o_m}\,(p_2) - \mathrm{o_m}\,(p_1) + (\mathrm{m}\,(p_2) - \mathrm{m}\,(p_1)) \mathrm{~div~} \mu_\mathrm{m})$
R9	R5 & R8	$\Rightarrow$	$\Delta\,\mathrm{g_c}\,(\Delta p) = \Delta\,\mathrm{p_c}\,(\Delta p) - \mu_\mathrm{c} \times (\mathrm{o_m}\,(p_2) - \mathrm{o_m}\,(p_1) + (\mathrm{m}\,(p_2) - \mathrm{m}\,(p_1)) \mathrm{~div~} \mu_\mathrm{m})$
R10	R9, 69 & 76	$\Rightarrow$	$\Delta\,\mathrm{g_c}\,(\Delta p) = \Delta\,\mathrm{p_c}\,(\Delta p)$
			$-\mu_\mathrm{c} \times \left(\begin{array}{l} (\mathrm{p_m}\,(p_2) \mathrm{~div~} \mu_\mathrm{m}) \\ -\,(\mathrm{p_m}\,(p_1) \mathrm{~div~} \mu_\mathrm{m}) \\ +\,((\mathrm{p_m}\,(p_2) \mathrm{~mod~} \mu_\mathrm{m}) - (\mathrm{p_m}\,(p_1) \mathrm{~mod~} \mu_\mathrm{m})) \mathrm{~div~} \mu_\mathrm{m} \end{array}\right)$
R11	R10 & 55	$\Rightarrow$	$\Delta\,\mathrm{g_c}\,(\Delta p) = \Delta\,\mathrm{p_c}\,(\Delta p) - \mu_\mathrm{c} \times ((\mathrm{p_m}\,(p_2) - \mathrm{p_m}\,(p_1)) \mathrm{~div~} \mu_\mathrm{m})$
R12	R11 & R7	$\Rightarrow$	$\Delta\,\mathrm{g_c}\,(\Delta p) = \Delta\,\mathrm{p_c}\,(\Delta p) - \mu_\mathrm{c} \times (\Delta\,\mathrm{p_m}\,(\Delta p) \mathrm{~div~} \mu_\mathrm{m})$

Definition 281 (Definition of $\Delta\,\mathrm{g}\,(\Delta p)$) *If p_1 and p_2 are any two pitches in a pitch system ψ then*

$$\Delta p = \Delta\,\mathrm{p}\,(p_1, p_2) \Rightarrow \Delta\,\mathrm{g}\,(\Delta p) = \Delta\,\mathrm{g}\,(p_1, p_2)$$

Theorem 282 (Formula for $\Delta\,\mathrm{g}\,(\Delta p)$) *If Δp is a pitch interval in ψ then:*

$$\Delta\,\mathrm{g}\,(\Delta p) = [\Delta\,\mathrm{g_c}\,(\Delta p)\,, \Delta\,\mathrm{m}\,(\Delta p)]$$

Proof

R1	Let		$\Delta p = \Delta\,\mathrm{p}\,(p_1, p_2)$
R2	R1 & 281	$\Rightarrow$	$\Delta\,\mathrm{g}\,(\Delta p) = \Delta\,\mathrm{g}\,(p_1, p_2)$
R3	R2 & 258	$\Rightarrow$	$\Delta\,\mathrm{g}\,(\Delta p) = [\Delta\,\mathrm{g_c}\,(p_1, p_2)\,, \Delta\,\mathrm{m}\,(p_1, p_2)]$
R4	R1 & 279	$\Rightarrow$	$\Delta\,\mathrm{g_c}\,(p_1, p_2) = \Delta\,\mathrm{g_c}\,(\Delta p)$
R5	R1 & 275	$\Rightarrow$	$\Delta\,\mathrm{m}\,(p_1, p_2) = \Delta\,\mathrm{m}\,(\Delta p)$
R6	R3, R4 & R5	$\Rightarrow$	$\Delta\,\mathrm{g}\,(\Delta p) = [\Delta\,\mathrm{g_c}\,(\Delta p)\,, \Delta\,\mathrm{m}\,(\Delta p)]$

Deriving *MIPS* intervals from a chromatic pitch interval

Definition 283 (Definition of $\Delta\,\mathrm{f}\,(\Delta p_\mathrm{c})$) *If $p_{\mathrm{c},1}$ and $p_{\mathrm{c},2}$ are any two chromatic pitches in a pitch system ψ then*

$$\Delta p_\mathrm{c} = \Delta\,\mathrm{p_c}\,(p_{\mathrm{c},1}, p_{\mathrm{c},2}) \Rightarrow \Delta\,\mathrm{f}\,(\Delta p_\mathrm{c}) = \Delta\,\mathrm{f}\,(p_{\mathrm{c},1}, p_{\mathrm{c},2})$$

Theorem 284 (Formula for $\Delta\,\mathrm{f}\,(\Delta p_\mathrm{c})$) *If Δp_c is a chromatic pitch interval in the pitch system ψ then*

$$\Delta\,\mathrm{f}\,(\Delta p_\mathrm{c}) = 2^{\Delta p_\mathrm{c}/\mu_\mathrm{c}}$$

Proof

R1	Let		$\Delta p_\mathrm{c} = \Delta\,\mathrm{p_c}\,(p_{\mathrm{c},1}, p_{\mathrm{c},2})$
R2	R1 & 283	$\Rightarrow$	$\Delta\,\mathrm{f}\,(\Delta p_\mathrm{c}) = \Delta\,\mathrm{f}\,(p_{\mathrm{c},1}, p_{\mathrm{c},2})$
R3	R2 & 235	$\Rightarrow$	$\Delta\,\mathrm{f}\,(\Delta p_\mathrm{c}) = 2^{(p_{\mathrm{c},2} - p_{\mathrm{c},1})/\mu_\mathrm{c}}$
R4	R1 & 236	$\Rightarrow$	$\Delta p_\mathrm{c} = p_{\mathrm{c},2} - p_{\mathrm{c},1}$
R5	R3 & R4	$\Rightarrow$	$\Delta\,\mathrm{f}\,(\Delta p_\mathrm{c}) = 2^{\Delta p_\mathrm{c}/\mu_\mathrm{c}}$

Theorem 285 ($\Delta\,\mathrm{f}\,(\Delta\,\mathrm{p_c}\,(\Delta p)) = \Delta\,\mathrm{f}\,(\Delta p)$) *If Δp is a pitch interval in ψ then*

$$\Delta\,\mathrm{f}\,(\Delta\,\mathrm{p_c}\,(\Delta p)) = \Delta\,\mathrm{f}\,(\Delta p)$$

Proof

R1	284	$\Rightarrow$	$\Delta\,\mathrm{f}\,(\Delta\,\mathrm{p_c}\,(\Delta p)) = 2^{\Delta\,\mathrm{p_c}(\Delta p)/\mu_\mathrm{c}}$
R2	272	$\Rightarrow$	$\Delta\,\mathrm{f}\,(\Delta p) = 2^{\Delta\,\mathrm{p_c}(\Delta p)/\mu_\mathrm{c}}$
R3	R1 & R2	$\Rightarrow$	$\Delta\,\mathrm{f}\,(\Delta\,\mathrm{p_c}\,(\Delta p)) = \Delta\,\mathrm{f}\,(\Delta p)$

Definition 286 (Definition of $\Delta\,\mathrm{c}\,(\Delta p_{\mathrm{c}})$) *If $p_{\mathrm{c},1}$ and $p_{\mathrm{c},2}$ are any two chromatic pitches in a pitch system ψ then*

$$\Delta p_{\mathrm{c}} = \Delta\,\mathrm{p_c}\,(p_{\mathrm{c},1}, p_{\mathrm{c},2}) \Rightarrow \Delta\,\mathrm{c}\,(\Delta p_{\mathrm{c}}) = \Delta\,\mathrm{c}\,(p_{\mathrm{c},1}, p_{\mathrm{c},2})$$

Theorem 287 (Formula for $\Delta\,\mathrm{c}\,(\Delta p_{\mathrm{c}})$) *If Δp_{c} is a chromatic pitch interval in the pitch system*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

then

$$\Delta\,\mathrm{c}\,(\Delta p_{\mathrm{c}}) = \Delta p_{\mathrm{c}} \bmod \mu_{\mathrm{c}}$$

Proof

R1	Let		$\Delta p_{\mathrm{c}} = \Delta\,\mathrm{p_c}\,(p_{\mathrm{c},1}, p_{\mathrm{c},2})$
R2	R1 & 286	$\Rightarrow$	$\Delta\,\mathrm{c}\,(\Delta p_{\mathrm{c}}) = \Delta\,\mathrm{c}\,(p_{\mathrm{c},1}, p_{\mathrm{c},2})$
R3	R2 & 233	$\Rightarrow$	$\Delta\,\mathrm{c}\,(\Delta p_{\mathrm{c}}) = (p_{\mathrm{c},2} - p_{\mathrm{c},1}) \bmod \mu_{\mathrm{c}}$
R4	R1 & 236	$\Rightarrow$	$\Delta p_{\mathrm{c}} = p_{\mathrm{c},2} - p_{\mathrm{c},1}$
R5	R3 & R4	$\Rightarrow$	$\Delta\,\mathrm{c}\,(\Delta p_{\mathrm{c}}) = \Delta p_{\mathrm{c}} \bmod \mu_{\mathrm{c}}$

Theorem 288 ($\Delta\,\mathrm{c}\,(\Delta\,\mathrm{p_c}\,(\Delta p)) = \Delta\,\mathrm{c}\,(\Delta p)$) *If Δp is a pitch interval in ψ then*

$$\Delta\,\mathrm{c}\,(\Delta\,\mathrm{p_c}\,(\Delta p)) = \Delta\,\mathrm{c}\,(\Delta p)$$

Proof

R1	287	$\Rightarrow$	$\Delta\,\mathrm{c}\,(\Delta\,\mathrm{p_c}\,(\Delta p)) = \Delta\,\mathrm{p_c}\,(\Delta p) \bmod \mu_{\mathrm{c}}$
R2	274	$\Rightarrow$	$\Delta\,\mathrm{c}\,(\Delta p) = \Delta\,\mathrm{p_c}\,(\Delta p) \bmod \mu_{\mathrm{c}}$
R3	R1 & R2	$\Rightarrow$	$\Delta\,\mathrm{c}\,(\Delta\,\mathrm{p_c}\,(\Delta p)) = \Delta\,\mathrm{c}\,(\Delta p)$

Deriving *MIPS* intervals from a morphetic pitch interval

Definition 289 (Definition of $\Delta\,\mathrm{m}\,(\Delta p_{\mathrm{m}})$) *If $p_{\mathrm{m},1}$ and $p_{\mathrm{m},2}$ are any two morphetic pitches in a pitch system ψ then*

$$\Delta p_{\mathrm{m}} = \Delta\,\mathrm{p_m}\,(p_{\mathrm{m},1}, p_{\mathrm{m},2}) \Rightarrow \Delta\,\mathrm{m}\,(\Delta p_{\mathrm{m}}) = \Delta\,\mathrm{m}\,(p_{\mathrm{m},1}, p_{\mathrm{m},2})$$

Theorem 290 (Formula for $\Delta\,\mathrm{m}\,(\Delta p_{\mathrm{m}})$) *If Δp_{m} is a morphetic pitch interval in the pitch system*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

then

$$\Delta\,\mathrm{m}\,(\Delta p_{\mathrm{m}}) = \Delta p_{\mathrm{m}} \bmod \mu_{\mathrm{m}}$$

Proof

R1	Let		$\Delta p_{\mathrm{m}} = \Delta\,\mathrm{p_m}\,(p_{\mathrm{m},1}, p_{\mathrm{m},2})$
R2	R1 & 289	$\Rightarrow$	$\Delta\,\mathrm{m}\,(\Delta p_{\mathrm{m}}) = \Delta\,\mathrm{m}\,(p_{\mathrm{m},1}, p_{\mathrm{m},2})$
R3	R2 & 239	$\Rightarrow$	$\Delta\,\mathrm{m}\,(\Delta p_{\mathrm{m}}) = (p_{\mathrm{m},2} - p_{\mathrm{m},1}) \bmod \mu_{\mathrm{m}}$
R4	R1 & 240	$\Rightarrow$	$\Delta p_{\mathrm{m}} = p_{\mathrm{m},2} - p_{\mathrm{m},1}$
R5	R3 & R4	$\Rightarrow$	$\Delta\,\mathrm{m}\,(\Delta p_{\mathrm{m}}) = \Delta p_{\mathrm{m}} \bmod \mu_{\mathrm{m}}$

Theorem 291 ($\Delta\,\mathrm{m}\,(\Delta\,\mathrm{p_m}\,(\Delta p)) = \Delta\,\mathrm{m}\,(\Delta p)$) *If Δp is a pitch interval in ψ then*

$$\Delta\,\mathrm{m}\,(\Delta\,\mathrm{p_m}\,(\Delta p)) = \Delta\,\mathrm{m}\,(\Delta p)$$

Proof

R1	290	$\Rightarrow$	$\Delta\,\mathrm{m}\,(\Delta\,\mathrm{p_m}\,(\Delta p)) = \Delta\,\mathrm{p_m}\,(\Delta p) \bmod \mu_{\mathrm{m}}$
R2	276	$\Rightarrow$	$\Delta\,\mathrm{m}\,(\Delta p) = \Delta\,\mathrm{p_m}\,(\Delta p) \bmod \mu_{\mathrm{m}}$
R3	R1 & R2	$\Rightarrow$	$\Delta\,\mathrm{m}\,(\Delta\,\mathrm{p_m}\,(\Delta p)) = \Delta\,\mathrm{m}\,(\Delta p)$

Deriving *MIPS* intervals from a frequency interval

Definition 292 (Definition of $\Delta\,\mathrm{p_c}\,(\Delta f)$) *If f_1 and f_2 are any two frequencies in a pitch system ψ then*

$$\Delta f = \Delta\,\mathrm{f}\,(f_1, f_2) \Rightarrow \Delta\,\mathrm{p_c}\,(\Delta f) = \Delta\,\mathrm{p_c}\,(f_1, f_2)$$

Theorem 293 (Formula for $\Delta\,\mathrm{p_c}\,(\Delta f)$) *If Δf is a frequency interval in*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

then

$$\Delta\,\mathrm{p_c}\,(\Delta f) = \mu_{\mathrm{c}} \times \frac{\ln(\Delta f)}{\ln 2}$$

Proof

R1	Let		$\Delta f = \Delta\,\mathrm{f}\,(f_1, f_2)$
R2	R1 & 292	$\Rightarrow$	$\Delta\,\mathrm{p_c}\,(\Delta f) = \Delta\,\mathrm{p_c}\,(f_1, f_2)$
R3	245	$\Rightarrow$	$\Delta\,\mathrm{p_c}\,(f_1, f_2) = \mu_\mathrm{c} \times \frac{\ln(f_2/f_1)}{\ln 2}$
R4	242	$\Rightarrow$	$\Delta\,\mathrm{f}\,(f_1, f_2) = \frac{f_2}{f_1}$
R5	R1 & R4	$\Rightarrow$	$\Delta f = \frac{f_2}{f_1}$
R6	R3 & R5	$\Rightarrow$	$\Delta\,\mathrm{p_c}\,(f_1, f_2) = \mu_\mathrm{c} \times \frac{\ln(\Delta f)}{\ln 2}$
R7	R2 & R6	$\Rightarrow$	$\Delta\,\mathrm{p_c}\,(\Delta f) = \mu_\mathrm{c} \times \frac{\ln(\Delta f)}{\ln 2}$

Theorem 294 ($\Delta\,\mathrm{p_c}\,(\Delta\,\mathrm{f}\,(\Delta p)) = \Delta\,\mathrm{p_c}\,(\Delta p)$) *If Δp is a pitch interval in ψ then*

$$\Delta\,\mathrm{p_c}\,(\Delta\,\mathrm{f}\,(\Delta p)) = \Delta\,\mathrm{p_c}\,(\Delta p)$$

Proof

R1	293	$\Rightarrow$	$\Delta\,\mathrm{p_c}\,(\Delta\,\mathrm{f}\,(\Delta p)) = \mu_\mathrm{c} \times \frac{\ln(\Delta\,\mathrm{f}(\Delta p))}{\ln 2}$
R2	272	$\Rightarrow$	$\Delta\,\mathrm{f}\,(\Delta p) = 2^{\Delta\,\mathrm{p_c}(\Delta p)/\mu_\mathrm{c}}$
R3	R1 & R2	$\Rightarrow$	$\Delta\,\mathrm{p_c}\,(\Delta\,\mathrm{f}\,(\Delta p)) = \mu_\mathrm{c} \times \frac{\ln\left(2^{\Delta\,\mathrm{p_c}(\Delta p)/\mu_\mathrm{c}}\right)}{\ln 2}$
R4	R3 & 59	$\Rightarrow$	$\Delta\,\mathrm{p_c}\,(\Delta\,\mathrm{f}\,(\Delta p)) = \mu_\mathrm{c} \times \log_2\left(2^{\Delta\,\mathrm{p_c}(\Delta p)/\mu_\mathrm{c}}\right)$
			$= \mu_\mathrm{c} \times \frac{\Delta\,\mathrm{p_c}(\Delta p)}{\mu_\mathrm{c}}$
			$= \Delta\,\mathrm{p_c}\,(\Delta p)$

Definition 295 (Definition of $\Delta\,\mathrm{c}\,(\Delta f)$) *If f_1 and f_2 are any two frequencies in a pitch system ψ then*

$$\Delta f = \Delta\,\mathrm{f}\,(f_1, f_2) \Rightarrow \Delta\,\mathrm{c}\,(\Delta f) = \Delta\,\mathrm{c}\,(f_1, f_2)$$

Theorem 296 (Formula for $\Delta\,\mathrm{c}\,(\Delta f)$) *If Δf is a frequency interval in a pitch system ψ then*

$$\Delta\,\mathrm{c}\,(\Delta f) = \left(\mu_\mathrm{c} \times \frac{\ln(\Delta f)}{\ln 2}\right) \bmod \mu_\mathrm{c}$$

Proof

R1	Let		$\Delta f = \Delta\,\mathrm{f}\,(f_1, f_2)$
R2	R1 & 295	$\Rightarrow$	$\Delta\,\mathrm{c}\,(\Delta f) = \Delta\,\mathrm{c}\,(f_1, f_2)$
R3	247	$\Rightarrow$	$\Delta\,\mathrm{c}\,(f_1, f_2) = \left(\mu_\mathrm{c} \times \frac{\ln(f_2/f_1)}{\ln 2}\right) \bmod \mu_\mathrm{c}$
R4	R3 & R2	$\Rightarrow$	$\Delta\,\mathrm{c}\,(\Delta f) = \left(\mu_\mathrm{c} \times \frac{\ln(f_2/f_1)}{\ln 2}\right) \bmod \mu_\mathrm{c}$
R5	242	$\Rightarrow$	$\Delta\,\mathrm{f}\,(f_1, f_2) = f_2/f_1$
R6	R5 & R1	$\Rightarrow$	$\Delta f = f_2/f_1$
R7	R6 & R4	$\Rightarrow$	$\Delta\,\mathrm{c}\,(\Delta f) = \left(\mu_\mathrm{c} \times \frac{\ln(\Delta f)}{\ln 2}\right) \bmod \mu_\mathrm{c}$

Theorem 297 (Second formula for $\Delta\,\mathrm{c}\,(\Delta f)$) *If Δf is a frequency interval in a pitch system ψ then*

$$\Delta\,\mathrm{c}\,(\Delta f) = \mu_\mathrm{c} \times \left(\frac{\ln(\Delta f)}{\ln 2} - \mathrm{int}\left(\frac{\ln(\Delta f)}{\ln 2}\right)\right)$$

Proof

R1	296	$\Rightarrow$	$\Delta\,\mathrm{c}\,(\Delta f) = \left(\mu_\mathrm{c} \times \frac{\ln(\Delta f)}{\ln 2}\right) \bmod \mu_\mathrm{c}$
R2	R1 & 33	$\Rightarrow$	$\Delta\,\mathrm{c}\,(\Delta f) = \frac{\mu_\mathrm{c} \ln(\Delta f)}{\ln 2} - \mu_\mathrm{c} \times \mathrm{int}\left(\frac{\mu_\mathrm{c} \ln(\Delta f)}{\mu_\mathrm{c} \ln 2}\right)$
			$= \mu_\mathrm{c} \times \left(\frac{\ln(\Delta f)}{\ln 2} - \mathrm{int}\left(\frac{\ln \Delta f}{\ln 2}\right)\right)$

Theorem 298 ($\Delta\,\mathrm{c}\,(\Delta\,\mathrm{f}\,(\Delta p)) = \Delta\,\mathrm{c}\,(\Delta p)$) *If Δp is a pitch interval in ψ then*

$$\Delta\,\mathrm{c}\,(\Delta\,\mathrm{f}\,(\Delta p)) = \Delta\,\mathrm{c}\,(\Delta p)$$

Proof

R1	296	$\Rightarrow$	$\Delta\,\mathrm{c}\left(\Delta\,\mathrm{f}\left(\Delta p\right)\right)=\left(\mu_{\mathrm{c}}\times\frac{\ln(\Delta\,\mathrm{f}(\Delta p))}{\ln 2}\right)\bmod\mu_{\mathrm{c}}$
R2	272	$\Rightarrow$	$\Delta\,\mathrm{f}\left(\Delta p\right)=2^{\Delta\,\mathrm{p_c}(\Delta p)/\mu_{\mathrm{c}}}$
R3	R1 & R2	$\Rightarrow$	$\Delta\,\mathrm{c}\left(\Delta\,\mathrm{f}\left(\Delta p\right)\right)=\left(\mu_{\mathrm{c}}\times\frac{\ln\left(2^{\Delta\,\mathrm{p_c}(\Delta p)/\mu_{\mathrm{c}}}\right)}{\ln 2}\right)\bmod\mu_{\mathrm{c}}$
R4	R3 & 59	$\Rightarrow$	$\Delta\,\mathrm{c}\left(\Delta\,\mathrm{f}\left(\Delta p\right)\right)=\left(\mu_{\mathrm{c}}\times\log_2\left(2^{\Delta\,\mathrm{p_c}(\Delta p)/\mu_{\mathrm{c}}}\right)\right)\bmod\mu_{\mathrm{c}}$
			$=\left(\mu_{\mathrm{c}}\times\left(\Delta\,\mathrm{p_c}\left(\Delta p\right)/\mu_{\mathrm{c}}\right)\right)\bmod\mu_{\mathrm{c}}$
			$=\Delta\,\mathrm{p_c}\left(\Delta p\right)\bmod\mu_{\mathrm{c}}$
R5	274	$\Rightarrow$	$\Delta\,\mathrm{c}\left(\Delta p\right)=\Delta\,\mathrm{p_c}\left(\Delta p\right)\bmod\mu_{\mathrm{c}}$
R6	R4 & R5	$\Rightarrow$	$\Delta\,\mathrm{c}\left(\Delta\,\mathrm{f}\left(\Delta p\right)\right)=\Delta\,\mathrm{c}\left(\Delta p\right)$

Deriving *MIPS* intervals from a chromamorph interval

Definition 299 (Definition of $\Delta\,\mathrm{c}\left(\Delta q\right)$) *If q_1 and q_2 are any two chromamorphs in a pitch system ψ then*

$$\Delta q=\Delta\,\mathrm{q}\left(q_1,q_2\right)\Rightarrow\Delta\,\mathrm{c}\left(\Delta q\right)=\Delta\,\mathrm{c}\left(q_1,q_2\right)$$

Theorem 300 (Formula for $\Delta\,\mathrm{c}\left(\Delta q\right)$) *If Δq is a chromamorph interval in a pitch system ψ then*

$$\Delta q=[\Delta c,\Delta m]\Rightarrow\Delta\,\mathrm{c}\left(\Delta q\right)=\Delta c$$

Proof

R1	Let		$\Delta q=\Delta\,\mathrm{q}\left(q_1,q_2\right)$
R2	Let		$\Delta q=[\Delta c,\Delta m]$
R3	R1 & 299	$\Rightarrow$	$\Delta\,\mathrm{c}\left(\Delta q\right)=\Delta\,\mathrm{c}\left(q_1,q_2\right)$
R4	223	$\Rightarrow$	$\Delta\,\mathrm{q}\left(q_1,q_2\right)=\left[\Delta\,\mathrm{c}\left(q_1,q_2\right),\Delta\,\mathrm{m}\left(q_1,q_2\right)\right]$
R5	R3 & R4	$\Rightarrow$	$\Delta\,\mathrm{q}\left(q_1,q_2\right)=\left[\Delta\,\mathrm{c}\left(\Delta q\right),\Delta\,\mathrm{m}\left(q_1,q_2\right)\right]$
R6	R1 & R5	$\Rightarrow$	$\Delta q=\left[\Delta\,\mathrm{c}\left(\Delta q\right),\Delta\,\mathrm{m}\left(q_1,q_2\right)\right]$
R7	R2 & R6	$\Rightarrow$	$\Delta\,\mathrm{c}\left(\Delta q\right)=\Delta c$

Theorem 301 ($\Delta\,\mathrm{c}\left(\Delta\,\mathrm{q}\left(\Delta p\right)\right)=\Delta\,\mathrm{c}\left(\Delta p\right)$) *If Δp is a pitch interval in a pitch system ψ then*

$$\Delta\,\mathrm{c}\left(\Delta\,\mathrm{q}\left(\Delta p\right)\right)=\Delta\,\mathrm{c}\left(\Delta p\right)$$

Proof

R1	274	$\Rightarrow$	$\Delta\,\mathrm{c}\,(\Delta p) = \Delta\,\mathrm{p_c}\,(\Delta p) \bmod \mu_\mathrm{c}$
R2	278	$\Rightarrow$	$\Delta\,\mathrm{q}\,(\Delta p) = [\Delta\,\mathrm{c}\,(\Delta p)\,, \Delta\,\mathrm{m}\,(\Delta p)]$
R3	Let		$\Delta q = [\Delta c, \Delta m]$
R4	R3 & 300	$\Rightarrow$	$\Delta\,\mathrm{c}\,(\Delta q) = \Delta c$
R5	Let		$\Delta q = \Delta\,\mathrm{q}\,(\Delta p)$
R6	R4 & R5	$\Rightarrow$	$\Delta\,\mathrm{c}\,(\Delta\,\mathrm{q}\,(\Delta p)) = \Delta c$
R7	R2, R3 & R5	$\Rightarrow$	$\Delta c = \Delta\,\mathrm{c}\,(\Delta p)$
R8	R6 & R7	$\Rightarrow$	$\Delta\,\mathrm{c}\,(\Delta\,\mathrm{q}\,(\Delta p)) = \Delta\,\mathrm{c}\,(\Delta p)$

Definition 302 (Definition of $\Delta\,\mathrm{m}\,(\Delta q)$) *If q_1 and q_2 are any two chromamorphs in a pitch system ψ then*

$$\Delta q = \Delta\,\mathrm{q}\,(q_1, q_2) \Rightarrow \Delta\,\mathrm{m}\,(\Delta q) = \Delta\,\mathrm{m}\,(q_1, q_2)$$

Theorem 303 (Formula for $\Delta\,\mathrm{m}\,(\Delta q)$) *If Δq is a chromamorph interval in a pitch system ψ then*

$$\Delta q = [\Delta c, \Delta m] \Rightarrow \Delta\,\mathrm{m}\,(\Delta q) = \Delta m$$

Proof

R1	Let		$\Delta q = \Delta\,\mathrm{q}\,(q_1, q_2)$
R2	Let		$\Delta q = [\Delta c, \Delta m]$
R3	R1 & 302	$\Rightarrow$	$\Delta\,\mathrm{m}\,(\Delta q) = \Delta\,\mathrm{m}\,(q_1, q_2)$
R4	223	$\Rightarrow$	$\Delta\,\mathrm{q}\,(q_1, q_2) = [\Delta\,\mathrm{c}\,(q_1, q_2)\,, \Delta\,\mathrm{m}\,(q_1, q_2)]$
R5	R3 & R4	$\Rightarrow$	$\Delta\,\mathrm{q}\,(q_1, q_2) = [\Delta\,\mathrm{c}\,(q_1, q_2)\,, \Delta\,\mathrm{m}\,(\Delta q)]$
R6	R1 & R5	$\Rightarrow$	$\Delta q = [\Delta\,\mathrm{c}\,(q_1, q_2)\,, \Delta\,\mathrm{m}\,(\Delta q)]$
R7	R2 & R6	$\Rightarrow$	$\Delta\,\mathrm{m}\,(\Delta q) = \Delta m$

Theorem 304 ($\Delta\,\mathrm{m}\,(\Delta\,\mathrm{q}\,(\Delta p)) = \Delta\,\mathrm{m}\,(\Delta p)$) *If Δp is a pitch interval in a pitch system ψ then*

$$\Delta\,\mathrm{m}\,(\Delta\,\mathrm{q}\,(\Delta p)) = \Delta\,\mathrm{m}\,(\Delta p)$$

Proof

R1	276	$\Rightarrow$	$\Delta\,\mathrm{m}\,(\Delta p) = \Delta\,\mathrm{p_m}\,(\Delta p) \bmod \mu_\mathrm{m}$
R2	278	$\Rightarrow$	$\Delta\,\mathrm{q}\,(\Delta p) = [\Delta\,\mathrm{c}\,(\Delta p)\,, \Delta\,\mathrm{m}\,(\Delta p)]$
R3	Let		$\Delta q = [\Delta c, \Delta m]$
R4	R3 & 303	$\Rightarrow$	$\Delta\,\mathrm{m}\,(\Delta q) = \Delta m$
R5	Let		$\Delta q = \Delta\,\mathrm{q}\,(\Delta p)$
R6	R4 & R5	$\Rightarrow$	$\Delta\,\mathrm{m}\,(\Delta\,\mathrm{q}\,(\Delta p)) = \Delta m$
R7	R2, R3 & R5	$\Rightarrow$	$\Delta m = \Delta\,\mathrm{m}\,(\Delta p)$
R8	R6 & R7	$\Rightarrow$	$\Delta\,\mathrm{m}\,(\Delta\,\mathrm{q}\,(\Delta p)) = \Delta\,\mathrm{m}\,(\Delta p)$

Theorem 305 ($\Delta q = [\Delta\,\mathrm{c}\,(\Delta q)\,, \Delta\,\mathrm{m}\,(\Delta q)]$) *If Δq is a chromamorph interval in ψ then*

$$\Delta q = [\Delta\,\mathrm{c}\,(\Delta q)\,, \Delta\,\mathrm{m}\,(\Delta q)]$$

Proof

R1	Let		$\Delta q = [\Delta c, \Delta m]$
R2	R1 & 300	$\Rightarrow$	$\Delta\,\mathrm{c}\,(\Delta q) = \Delta c$
R3	R1 & 303	$\Rightarrow$	$\Delta\,\mathrm{m}\,(\Delta q) = \Delta m$
R4	R1, R2 & R4	$\Rightarrow$	$\Delta q = [\Delta\,\mathrm{c}\,(\Delta q)\,, \Delta\,\mathrm{m}\,(\Delta q)]$

Deriving *MIPS* intervals from a chromatic genus interval

Definition 306 (Definition of $\Delta\,\mathrm{c}\,(\Delta g_\mathrm{c})$) *If g_1 and g_2 are two genera in a pitch system ψ then*

$$\Delta g_\mathrm{c} = \Delta\,\mathrm{g_c}\,(g_1, g_2) \Rightarrow \Delta\,\mathrm{c}\,(\Delta g_\mathrm{c}) = \Delta\,\mathrm{c}\,(g_1, g_2)$$

Theorem 307 (Formula for $\Delta\,\mathrm{c}\,(\Delta g_\mathrm{c})$) *If Δg_c is a chromatic genus interval in a pitch system*

$$\psi = [\mu_\mathrm{c}, \mu_\mathrm{m}, f_0, p_{\mathrm{c},0}]$$

then

$$\Delta\,\mathrm{c}\,(\Delta g_\mathrm{c}) = \Delta g_\mathrm{c} \bmod \mu_\mathrm{c}$$

Proof

R1	Let		$\Delta g_{\mathrm{c}} = \Delta\,\mathrm{g_c}\,(g_1, g_2)$
R2	R1 & 306	$\Rightarrow$	$\Delta\,\mathrm{c}\,(\Delta g_{\mathrm{c}}) = \Delta\,\mathrm{c}\,(g_1, g_2)$
R3	R2 & 227	$\Rightarrow$	$\Delta\,\mathrm{c}\,(\Delta g_{\mathrm{c}}) = (\mathrm{g_c}\,(g_2) - \mathrm{g_c}\,(g_1)) \bmod \mu_{\mathrm{c}}$
R4	R1 & 230	$\Rightarrow$	$\Delta g_{\mathrm{c}} = \mathrm{g_c}\,(g_2) - \mathrm{g_c}\,(g_1) - \mu_{\mathrm{c}} \times ((\mathrm{m}\,(g_2) - \mathrm{m}\,(g_1)) \operatorname{div} \mu_{\mathrm{m}})$
R5	48	$\Rightarrow$	$((\mathrm{m}\,(g_2) - \mathrm{m}\,(g_1)) \operatorname{div} \mu_{\mathrm{m}})$ is an integer
R6	R5 & 37	$\Rightarrow$	$(\mathrm{g_c}\,(g_2) - \mathrm{g_c}\,(g_1) - \mu_{\mathrm{c}} \times ((\mathrm{m}\,(g_2) - \mathrm{m}\,(g_1)) \operatorname{div} \mu_{\mathrm{m}})) \bmod \mu_{\mathrm{c}}$
			$= (\mathrm{g_c}\,(g_2) - \mathrm{g_c}\,(g_1)) \bmod \mu_{\mathrm{c}}$
R7	R6, R4 & R3	$\Rightarrow$	$\Delta\,\mathrm{c}\,(\Delta g_{\mathrm{c}}) = \Delta g_{\mathrm{c}} \bmod \mu_{\mathrm{c}}$

Theorem 308 ($\Delta\,\mathrm{c}\,(\Delta\,\mathrm{g_c}\,(\Delta p)) = \Delta\,\mathrm{c}\,(\Delta p)$) *If Δp is a pitch interval in a pitch system ψ then*

$$\Delta\,\mathrm{c}\,(\Delta\,\mathrm{g_c}\,(\Delta p)) = \Delta\,\mathrm{c}\,(\Delta p)$$

Proof

R1	274	$\Rightarrow$	$\Delta\,\mathrm{c}\,(\Delta p) = \Delta\,\mathrm{p_c}\,(\Delta p) \bmod \mu_{\mathrm{c}}$
R2	307	$\Rightarrow$	$\Delta\,\mathrm{c}\,(\Delta g_{\mathrm{c}}) = \Delta g_{\mathrm{c}} \bmod \mu_{\mathrm{c}}$
R3	280	$\Rightarrow$	$\Delta\,\mathrm{g_c}\,(\Delta p) = \Delta\,\mathrm{p_c}\,(\Delta p) - \mu_{\mathrm{c}} \times (\Delta\,\mathrm{p_m}\,(\Delta p) \operatorname{div} \mu_{\mathrm{m}})$
R4	Let		$\Delta\,\mathrm{g_c}\,(\Delta p) = \Delta g_{\mathrm{c}}$
R5	R4 & R2	$\Rightarrow$	$\Delta\,\mathrm{c}\,(\Delta\,\mathrm{g_c}\,(\Delta p)) = \Delta\,\mathrm{g_c}\,(\Delta p) \bmod \mu_{\mathrm{c}}$
R6	R3 & R5	$\Rightarrow$	$\Delta\,\mathrm{c}\,(\Delta\,\mathrm{g_c}\,(\Delta p)) = (\Delta\,\mathrm{p_c}\,(\Delta p) - \mu_{\mathrm{c}} \times (\Delta\,\mathrm{p_m}\,(\Delta p) \operatorname{div} \mu_{\mathrm{m}})) \bmod \mu_{\mathrm{c}}$
R7	48	$\Rightarrow$	$(\Delta\,\mathrm{p_m}\,(\Delta p) \operatorname{div} \mu_{\mathrm{m}})$ is an integer
R8	R7, R6 & 37	$\Rightarrow$	$\Delta\,\mathrm{c}\,(\Delta\,\mathrm{g_c}\,(\Delta p)) = \Delta\,\mathrm{p_c}\,(\Delta p) \bmod \mu_{\mathrm{c}}$
R9	R1 & R8	$\Rightarrow$	$\Delta\,\mathrm{c}\,(\Delta\,\mathrm{g_c}\,(\Delta p)) = \Delta\,\mathrm{c}\,(\Delta p)$

Deriving *MIPS* intervals from a genus interval

Definition 309 (Chromatic genus interval of a genus interval) *If g_1 and g_2 are two genera in a pitch system ψ then*

$$\Delta g = \Delta\,\mathrm{g}\,(g_1, g_2) \Rightarrow \Delta\,\mathrm{g_c}\,(\Delta g) = \Delta\,\mathrm{g_c}\,(g_1, g_2)$$

Theorem 310 (Formula for chromatic genus interval of a genus interval) *If Δg is a genus interval in a pitch system ψ then*

$$\Delta g = [\Delta g_{\mathrm{c}}, \Delta m] \Rightarrow \Delta\,\mathrm{g_c}\,(\Delta g) = \Delta g_{\mathrm{c}}$$

Proof

R1	Let		$\Delta g = \Delta\,\mathrm{g}\,(g_1, g_2)$
R2	R1 & 309	$\Rightarrow$	$\Delta\,\mathrm{g_c}\,(\Delta g) = \Delta\,\mathrm{g_c}\,(g_1, g_2)$
R3	R1 & 231	$\Rightarrow$	$\Delta g = [\Delta\,\mathrm{g_c}\,(g_1, g_2)\,, \Delta\,\mathrm{m}\,(g_1, g_2)]$
R4	Let		$\Delta g = [\Delta g_{\mathrm{c}}, \Delta m]$
R5	R3 & R4	$\Rightarrow$	$\Delta g_{\mathrm{c}} = \Delta\,\mathrm{g_c}\,(g_1, g_2)$
R6	R5 & R2	$\Rightarrow$	$\Delta\,\mathrm{g_c}\,(\Delta g) = \Delta g_{\mathrm{c}}$

Theorem 311 ($\Delta\,\mathrm{g_c}\,(\Delta\,\mathrm{g}\,(\Delta p)) = \Delta\,\mathrm{g_c}\,(\Delta p)$) *If Δp is a pitch interval in a pitch system ψ then*

$$\Delta\,\mathrm{g_c}\,(\Delta\,\mathrm{g}\,(\Delta p)) = \Delta\,\mathrm{g_c}\,(\Delta p)$$

Proof

R1	Let		$\Delta g = [\Delta g_{\mathrm{c}}, \Delta m]$
R2	R1 & 310	$\Rightarrow$	$\Delta\,\mathrm{g_c}\,(\Delta g) = \Delta g_{\mathrm{c}}$
R3	282	$\Rightarrow$	$\Delta\,\mathrm{g}\,(\Delta p) = [\Delta\,\mathrm{g_c}\,(\Delta p)\,, \Delta\,\mathrm{m}\,(\Delta p)]$
R4	Let		$\Delta g = \Delta\,\mathrm{g}\,(\Delta p)$
R5	R1, R3 & R4	$\Rightarrow$	$\Delta g_{\mathrm{c}} = \Delta\,\mathrm{g_c}\,(\Delta p)$
R6	R2 & R4	$\Rightarrow$	$\Delta\,\mathrm{g_c}\,(\Delta\,\mathrm{g}\,(\Delta p)) = \Delta g_{\mathrm{c}}$
R7	R5 & R6	$\Rightarrow$	$\Delta\,\mathrm{g_c}\,(\Delta\,\mathrm{g}\,(\Delta p)) = \Delta\,\mathrm{g_c}\,(\Delta p)$

Definition 312 (Definition of $\Delta\,\mathrm{c}\,(\Delta g)$) *If g_1 and g_2 are two genera in a pitch system ψ then*

$$\Delta g = \Delta\,\mathrm{g}\,(g_1, g_2) \Rightarrow \Delta\,\mathrm{c}\,(\Delta g) = \Delta\,\mathrm{c}\,(g_1, g_2)$$

Theorem 313 (Formula for $\Delta\,\mathrm{c}\,(\Delta g)$) *If Δg is a genus interval in a pitch system ψ then*

$$\Delta\,\mathrm{c}\,(\Delta g) = \Delta\,\mathrm{g_c}\,(\Delta g) \bmod \mu_{\mathrm{c}}$$

Proof

R1	Let		$\Delta g = \Delta\,\mathrm{g}\,(g_1, g_2)$
R2	R1 & 312	$\Rightarrow$	$\Delta\,\mathrm{c}\,(\Delta g) = \Delta\,\mathrm{c}\,(g_1, g_2)$
R3	R1 & 309	$\Rightarrow$	$\Delta\,\mathrm{g_c}\,(\Delta g) = \Delta\,\mathrm{g_c}\,(g_1, g_2)$
R4	R2 & 227	$\Rightarrow$	$\Delta\,\mathrm{c}\,(\Delta g) = (\mathrm{g_c}\,(g_2) - \mathrm{g_c}\,(g_1)) \bmod \mu_\mathrm{c}$
R5	R3 & 230	$\Rightarrow$	$\Delta\,\mathrm{g_c}\,(\Delta g) = \mathrm{g_c}\,(g_2) - \mathrm{g_c}\,(g_1) - \mu_\mathrm{c} \times ((\mathrm{m}\,(g_2) - \mathrm{m}\,(g_1)) \operatorname{div} \mu_\mathrm{m})$
R6	48	$\Rightarrow$	$((\mathrm{m}\,(g_2) - \mathrm{m}\,(g_1)) \operatorname{div} \mu_\mathrm{m})$ is an integer
R7	R6 & 37	$\Rightarrow$	$(\mathrm{g_c}\,(g_2) - \mathrm{g_c}\,(g_1) - \mu_\mathrm{c} \times ((\mathrm{m}\,(g_2) - \mathrm{m}\,(g_1)) \operatorname{div} \mu_\mathrm{m})) \bmod \mu_\mathrm{c}$
			$= (\mathrm{g_c}\,(g_2) - \mathrm{g_c}\,(g_1)) \bmod \mu_\mathrm{c}$
R8	R7 & R5	$\Rightarrow$	$\Delta\,\mathrm{g_c}\,(\Delta g) \bmod \mu_\mathrm{c} = (\mathrm{g_c}\,(g_2) - \mathrm{g_c}\,(g_1)) \bmod \mu_\mathrm{c}$
R9	R4 & R8	$\Rightarrow$	$\Delta\,\mathrm{c}\,(\Delta g) = \Delta\,\mathrm{g_c}\,(\Delta g) \bmod \mu_\mathrm{c}$

Theorem 314 ($\Delta\,\mathrm{c}\,(\Delta\,\mathrm{g}\,(\Delta p)) = \Delta\,\mathrm{c}\,(\Delta p)$) *If Δp is a pitch interval in a pitch system ψ then*

$$\Delta\,\mathrm{c}\,(\Delta\,\mathrm{g}\,(\Delta p)) = \Delta\,\mathrm{c}\,(\Delta p)$$

Proof

R1	Let		$\Delta g = \Delta\,\mathrm{g}\,(\Delta p)$
R2	R1 & 313	$\Rightarrow$	$\Delta\,\mathrm{c}\,(\Delta\,\mathrm{g}\,(\Delta p)) = \Delta\,\mathrm{g_c}\,(\Delta\,\mathrm{g}\,(\Delta p)) \bmod \mu_\mathrm{c}$
R3	R2 & 311	$\Rightarrow$	$\Delta\,\mathrm{c}\,(\Delta\,\mathrm{g}\,(\Delta p)) = \Delta\,\mathrm{g_c}\,(\Delta p) \bmod \mu_\mathrm{c}$
R4	Let		$\Delta g_\mathrm{c} = \Delta\,\mathrm{g_c}\,(\Delta p)$
R5	R4 & 307	$\Rightarrow$	$\Delta\,\mathrm{c}\,(\Delta\,\mathrm{g_c}\,(\Delta p)) = \Delta\,\mathrm{g_c}\,(\Delta p) \bmod \mu_\mathrm{c}$
R6	R3 & R5	$\Rightarrow$	$\Delta\,\mathrm{c}\,(\Delta\,\mathrm{g}\,(\Delta p)) = \Delta\,\mathrm{c}\,(\Delta\,\mathrm{g_c}\,(\Delta p))$
R7	R6 & 308	$\Rightarrow$	$\Delta\,\mathrm{c}\,(\Delta\,\mathrm{g}\,(\Delta p)) = \Delta\,\mathrm{c}\,(\Delta p)$

Definition 315 (Morph interval of a genus interval) *If g_1 and g_2 are two genera in a pitch system ψ then*

$$\Delta g = \Delta\,\mathrm{g}\,(g_1, g_2) \Rightarrow \Delta\,\mathrm{m}\,(\Delta g) = \Delta\,\mathrm{m}\,(g_1, g_2)$$

Theorem 316 (Formula for morph interval of a genus interval) *If Δg is a genus interval in a pitch system ψ then*

$$\Delta g = [\Delta g_{\mathrm{c}}, \Delta m] \Rightarrow \Delta\,\mathrm{m}\,(\Delta g) = \Delta m$$

Proof

R1	Let		$\Delta g = \Delta\,\mathrm{g}\,(g_1, g_2)$
R2	R1 & 315	$\Rightarrow$	$\Delta\,\mathrm{m}\,(\Delta g) = \Delta\,\mathrm{m}\,(g_1, g_2)$
R3	R1 & 231	$\Rightarrow$	$\Delta g = [\Delta\,\mathrm{g_c}\,(g_1, g_2)\,, \Delta\,\mathrm{m}\,(g_1, g_2)]$
R4	Let		$\Delta g = [\Delta g_{\mathrm{c}}, \Delta m]$
R5	R3 & R4	$\Rightarrow$	$\Delta m = \Delta\,\mathrm{m}\,(g_1, g_2)$
R6	R5 & R2	$\Rightarrow$	$\Delta\,\mathrm{m}\,(\Delta g) = \Delta m$

Theorem 317 ($\Delta\,\mathrm{m}\,(\Delta\,\mathrm{g}\,(\Delta p)) = \Delta\,\mathrm{m}\,(\Delta p)$) *If Δp is a pitch interval in a pitch system ψ then*

$$\Delta\,\mathrm{m}\,(\Delta\,\mathrm{g}\,(\Delta p)) = \Delta\,\mathrm{m}\,(\Delta p)$$

Proof

R1	Let		$\Delta g = [\Delta g_{\mathrm{c}}, \Delta m]$
R2	R1 & 316	$\Rightarrow$	$\Delta\,\mathrm{m}\,(\Delta g) = \Delta m$
R3	282	$\Rightarrow$	$\Delta\,\mathrm{g}\,(\Delta p) = [\Delta\,\mathrm{g_c}\,(\Delta p)\,, \Delta\,\mathrm{m}\,(\Delta p)]$
R4	Let		$\Delta g = \Delta\,\mathrm{g}\,(\Delta p)$
R5	R1, R3 & R4	$\Rightarrow$	$\Delta m = \Delta\,\mathrm{m}\,(\Delta p)$
R6	R2 & R4	$\Rightarrow$	$\Delta\,\mathrm{m}\,(\Delta\,\mathrm{g}\,(\Delta p)) = \Delta m$
R7	R5 & R6	$\Rightarrow$	$\Delta\,\mathrm{m}\,(\Delta\,\mathrm{g}\,(\Delta p)) = \Delta\,\mathrm{m}\,(\Delta p)$

Theorem 318 *If Δg is a genus interval in ψ then*

$$\Delta g = [\Delta\,\mathrm{g_c}\,(\Delta g)\,, \Delta\,\mathrm{m}\,(\Delta g)]$$

Proof

R1	Let		$\Delta g = [\Delta g_{\mathrm{c}}, \Delta m]$
R2	R1 & 310	$\Rightarrow$	$\Delta\,\mathrm{g_c}\left(\Delta g\right) = \Delta g_{\mathrm{c}}$
R3	R1 & 316	$\Rightarrow$	$\Delta\,\mathrm{m}\left(\Delta g\right) = \Delta m$
R4	R1, R2 & R3 $\Rightarrow$	$\Delta g = \left[\Delta\,\mathrm{g_c}\left(\Delta g\right), \Delta\,\mathrm{m}\left(\Delta g\right)\right]$	

Definition 319 (Definition of $\Delta\,\mathrm{q}\left(\Delta g\right)$) *If g_1 and g_2 are two genera in a pitch system ψ then*

$$\Delta g = \Delta\,\mathrm{g}\left(g_1, g_2\right) \Rightarrow \Delta\,\mathrm{q}\left(\Delta g\right) = \Delta\,\mathrm{q}\left(g_1, g_2\right)$$

Theorem 320 (Formula for $\Delta\,\mathrm{q}\left(\Delta g\right)$) *If Δg is a genus interval in a pitch system ψ then*

$$\Delta\,\mathrm{q}\left(\Delta g\right) = \left[\Delta\,\mathrm{c}\left(\Delta g\right), \Delta\,\mathrm{m}\left(\Delta g\right)\right]$$

Proof

R1	Let		$\Delta g = \Delta\,\mathrm{g}\,(g_1, g_2)$
R2	R1 & 319	$\Rightarrow$	$\Delta\,\mathrm{q}\,(\Delta g) = \Delta\,\mathrm{q}\,(g_1, g_2)$
R3	R2 & 229	$\Rightarrow$	$\Delta\,\mathrm{q}\,(\Delta g) = \Delta\,\mathrm{q}\,(\mathrm{q}\,(g_1)\,, \mathrm{q}\,(g_2))$
R4	R3 & 223	$\Rightarrow$	$\Delta\,\mathrm{q}\,(\Delta g) = [\Delta\,\mathrm{c}\,(\mathrm{q}\,(g_1)\,, \mathrm{q}\,(g_2))\,, \Delta\,\mathrm{m}\,(\mathrm{q}\,(g_1)\,, \mathrm{q}\,(g_2))]$
R5	R4, 221 & 222	$\Rightarrow$	$\Delta\,\mathrm{q}\,(\Delta g) = [\Delta\,\mathrm{c}\,(\mathrm{c}\,(\mathrm{q}\,(g_1))\,, \mathrm{c}\,(\mathrm{q}\,(g_2)))\,, \Delta\,\mathrm{m}\,(\mathrm{m}\,(\mathrm{q}\,(g_1))\,, \mathrm{m}\,(\mathrm{q}\,(g_2)))]$
R6	Let		$g_1 = \mathrm{g}\,(p_1)$ and $g_2 = \mathrm{g}\,(p_2)$
R7	R5 & R6	$\Rightarrow$	$\Delta\,\mathrm{q}\,(\Delta g) = [\Delta\,\mathrm{c}\,(\mathrm{c}\,(\mathrm{q}\,(\mathrm{g}\,(p_1)))\,, \mathrm{c}\,(\mathrm{q}\,(\mathrm{g}\,(p_2))))\,, \Delta\,\mathrm{m}\,(\mathrm{m}\,(\mathrm{q}\,(\mathrm{g}\,(p_1)))\,, \mathrm{m}\,(\mathrm{q}\,(\mathrm{g}\,(p_2))))]$
R8	R7 & 121	$\Rightarrow$	$\Delta\,\mathrm{q}\,(\Delta g) = [\Delta\,\mathrm{c}\,(\mathrm{c}\,(\mathrm{q}\,(p_1))\,, \mathrm{c}\,(\mathrm{q}\,(p_2)))\,, \Delta\,\mathrm{m}\,(\mathrm{m}\,(\mathrm{q}\,(p_1))\,, \mathrm{m}\,(\mathrm{q}\,(p_2)))]$
R9	R8, 107 & 105	$\Rightarrow$	$\Delta\,\mathrm{q}\,(\Delta g) = [\Delta\,\mathrm{c}\,(\mathrm{c}\,(p_1)\,, \mathrm{c}\,(p_2))\,, \Delta\,\mathrm{m}\,(\mathrm{m}\,(p_1)\,, \mathrm{m}\,(p_2))]$
R10	R1 & 312	$\Rightarrow$	$\Delta\,\mathrm{c}\,(\Delta g) = \Delta\,\mathrm{c}\,(g_1, g_2)$
R11	R10 & 226	$\Rightarrow$	$\Delta\,\mathrm{c}\,(\Delta g) = \Delta\,\mathrm{c}\,(\mathrm{c}\,(g_1)\,, \mathrm{c}\,(g_2))$
R12	R11 & R6	$\Rightarrow$	$\Delta\,\mathrm{c}\,(\Delta g) = \Delta\,\mathrm{c}\,(\mathrm{c}\,(\mathrm{g}\,(p_1))\,, \mathrm{c}\,(\mathrm{g}\,(p_2)))$
R13	R12 & 119	$\Rightarrow$	$\Delta\,\mathrm{c}\,(\Delta g) = \Delta\,\mathrm{c}\,(\mathrm{c}\,(p_1)\,, \mathrm{c}\,(p_2))$
R14	R1 & 315	$\Rightarrow$	$\Delta\,\mathrm{m}\,(\Delta g) = \Delta\,\mathrm{m}\,(g_1, g_2)$
R15	R14 & 228	$\Rightarrow$	$\Delta\,\mathrm{m}\,(\Delta g) = \Delta\,\mathrm{m}\,(\mathrm{m}\,(g_1)\,, \mathrm{m}\,(g_2))$
R16	R15 & R6	$\Rightarrow$	$\Delta\,\mathrm{m}\,(\Delta g) = \Delta\,\mathrm{m}\,(\mathrm{m}\,(\mathrm{g}\,(p_1))\,, \mathrm{m}\,(\mathrm{g}\,(p_2)))$
R17	R16 & 116	$\Rightarrow$	$\Delta\,\mathrm{m}\,(\Delta g) = \Delta\,\mathrm{m}\,(\mathrm{m}\,(p_1)\,, \mathrm{m}\,(p_2))$
R18	R9, R13 & R17	$\Rightarrow$	$\Delta\,\mathrm{q}\,(\Delta g) = [\Delta\,\mathrm{c}\,(\Delta g)\,, \Delta\,\mathrm{m}\,(\Delta g)]$

Theorem 321 ($\Delta\,\mathrm{q}\,(\Delta\,\mathrm{g}\,(\Delta p)) = \Delta\,\mathrm{q}\,(\Delta p)$) *If Δp is a pitch interval in a pitch system ψ then*

$$\Delta\,\mathrm{q}\,(\Delta\,\mathrm{g}\,(\Delta p)) = \Delta\,\mathrm{q}\,(\Delta p)$$

Proof

R1	278	$\Rightarrow$	$\Delta\,\mathrm{q}\,(\Delta p) = [\Delta\,\mathrm{c}\,(\Delta p)\,, \Delta\,\mathrm{m}\,(\Delta p)]$
R2	320	$\Rightarrow$	$\Delta\,\mathrm{q}\,(\Delta g) = [\Delta\,\mathrm{c}\,(\Delta g)\,, \Delta\,\mathrm{m}\,(\Delta g)]$
R3	Let		$\Delta\,\mathrm{g}\,(\Delta p) = \Delta g$
R4	R2 & R3	$\Rightarrow$	$\Delta\,\mathrm{q}\,(\Delta\,\mathrm{g}\,(\Delta p)) = [\Delta\,\mathrm{c}\,(\Delta\,\mathrm{g}\,(\Delta p))\,, \Delta\,\mathrm{m}\,(\Delta\,\mathrm{g}\,(\Delta p))]$
R5	314	$\Rightarrow$	$\Delta\,\mathrm{c}\,(\Delta\,\mathrm{g}\,(\Delta p)) = \Delta\,\mathrm{c}\,(\Delta p)$
R6	317	$\Rightarrow$	$\Delta\,\mathrm{m}\,(\Delta\,\mathrm{g}\,(\Delta p)) = \Delta\,\mathrm{m}\,(\Delta p)$
R7	R4, R5 & R6	$\Rightarrow$	$\Delta\,\mathrm{q}\,(\Delta\,\mathrm{g}\,(\Delta p)) = [\Delta\,\mathrm{c}\,(\Delta p)\,, \Delta\,\mathrm{m}\,(\Delta p)]$
R8	R7 & R1	$\Rightarrow$	$\Delta\,\mathrm{q}\,(\Delta\,\mathrm{g}\,(\Delta p)) = \Delta\,\mathrm{q}\,(\Delta p)$

4.4.3 Equivalence relations between *MIPS* intervals

Equivalence relations between pitch intervals

Definition 322 **($\Delta p_1 \equiv_{\Delta\,\mathrm{p_c}} \Delta p_2$)** *Two pitch intervals Δp_1 and Δp_2 are* chromatic pitch interval equivalent *if and only if*

$$\Delta\,\mathrm{p_c}\,(\Delta p_1) = \Delta\,\mathrm{p_c}\,(\Delta p_2)$$

The fact that two pitch intervals are chromatic pitch interval equivalent is denoted as follows:

$$\Delta p_1 \equiv_{\Delta\,\mathrm{p_c}} \Delta p_2$$

Definition 323 **($\Delta p_1 \equiv_{\Delta\,\mathrm{p_m}} \Delta p_2$)** *Two pitch intervals Δp_1 and Δp_2 are* morphetic pitch interval equivalent *if and only if*

$$\Delta\,\mathrm{p_m}\,(\Delta p_1) = \Delta\,\mathrm{p_m}\,(\Delta p_2)$$

The fact that two pitch intervals are morphetic pitch interval equivalent is denoted as follows:

$$\Delta p_1 \equiv_{\Delta\,\mathrm{p_m}} \Delta p_2$$

Definition 324 **($\Delta p_1 \equiv_{\Delta\,\mathrm{f}} \Delta p_2$)** *Two pitch intervals Δp_1 and Δp_2 are* frequency interval equivalent *if and only if*

$$\Delta\,\mathrm{f}\,(\Delta p_1) = \Delta\,\mathrm{f}\,(\Delta p_2)$$

The fact that two pitch intervals are frequency interval equivalent is denoted as follows:

$$\Delta p_1 \equiv_{\Delta\,\mathrm{f}} \Delta p_2$$

Definition 325 **($\Delta p_1 \equiv_{\Delta\,\mathrm{c}} \Delta p_2$)** *Two pitch intervals Δp_1 and Δp_2 are* chroma interval equivalent *if and only if*

$$\Delta\,\mathrm{c}\,(\Delta p_1) = \Delta\,\mathrm{c}\,(\Delta p_2)$$

The fact that two pitch intervals are chroma interval equivalent is denoted as follows:

$$\Delta p_1 \equiv_{\Delta\,\mathrm{c}} \Delta p_2$$

Definition 326 **($\Delta p_1 \equiv_{\Delta\,\mathrm{m}} \Delta p_2$)** *Two pitch intervals Δp_1 and Δp_2 are* morph interval equivalent *if and only if*

$$\Delta\,\mathrm{m}\,(\Delta p_1) = \Delta\,\mathrm{m}\,(\Delta p_2)$$

The fact that two pitch intervals are morph interval equivalent is denoted as follows:

$$\Delta p_1 \equiv_{\Delta\,\mathrm{m}} \Delta p_2$$

Definition 327 **($\Delta p_1 \equiv_{\Delta\,\mathrm{q}} \Delta p_2$)** *Two pitch intervals Δp_1 and Δp_2 are* chromamorph interval equivalent *if and only if*

$$\Delta\,\mathrm{q}\,(\Delta p_1) = \Delta\,\mathrm{q}\,(\Delta p_2)$$

The fact that two pitch intervals are chromamorph interval equivalent is denoted as follows:

$$\Delta p_1 \equiv_{\Delta\,\mathrm{q}} \Delta p_2$$

Definition 328 **($\Delta p_1 \equiv_{\Delta\,\mathrm{g_c}} \Delta p_2$)** *Two pitch intervals Δp_1 and Δp_2 are* chromatic genus interval equivalent *if and only if*

$$\Delta\,\mathrm{g_c}\,(\Delta p_1) = \Delta\,\mathrm{g_c}\,(\Delta p_2)$$

The fact that two pitch intervals are chromatic genus interval equivalent is denoted as follows:

$$\Delta p_1 \equiv_{\Delta\,\mathrm{g_c}} \Delta p_2$$

Definition 329 **($\Delta p_1 \equiv_{\Delta\,\mathrm{g}} \Delta p_2$)** *Two pitch intervals Δp_1 and Δp_2 are* genus interval equivalent *if and only if*

$$\Delta\,\mathrm{g}\,(\Delta p_1) = \Delta\,\mathrm{g}\,(\Delta p_2)$$

The fact that two pitch intervals are genus interval equivalent is denoted as follows:

$$\Delta p_1 \equiv_{\Delta\,\mathrm{g}} \Delta p_2$$

Equivalence relations between chromatic pitch intervals

Definition 330 **($\Delta p_{\mathrm{c},1} \equiv_{\Delta\,\mathrm{f}} \Delta p_{\mathrm{c},2}$)** *Two chromatic pitch intervals $\Delta p_{\mathrm{c},1}$ and $\Delta p_{\mathrm{c},2}$ are* frequency interval equivalent *if and only if*

$$\Delta\,\mathrm{f}\,(\Delta p_{\mathrm{c},1}) = \Delta\,\mathrm{f}\,(\Delta p_{\mathrm{c},2})$$

The fact that two chromatic pitch intervals are frequency interval equivalent is denoted as follows:

$$\Delta p_{\mathrm{c},1} \equiv_{\Delta\,\mathrm{f}} \Delta p_{\mathrm{c},2}$$

Definition 331 **($\Delta p_{\mathrm{c},1} \equiv_{\Delta\,\mathrm{c}} \Delta p_{\mathrm{c},2}$)** *Two chromatic pitch intervals $\Delta p_{\mathrm{c},1}$ and $\Delta p_{\mathrm{c},2}$ are* chroma interval equivalent *if and only if*

$$\Delta\,\mathrm{c}\,(\Delta p_{\mathrm{c},1}) = \Delta\,\mathrm{c}\,(\Delta p_{\mathrm{c},2})$$

The fact that two chromatic pitch intervals are chroma interval equivalent is denoted as follows:

$$\Delta p_{\mathrm{c},1} \equiv_{\Delta\,\mathrm{c}} \Delta p_{\mathrm{c},2}$$

Equivalence relations between morphetic pitch intervals

Definition 332 **($\Delta p_{\mathrm{m},1} \equiv_{\Delta\,\mathrm{m}} \Delta p_{\mathrm{m},2}$)** *Two morphetic pitch intervals $\Delta p_{\mathrm{m},1}$ and $\Delta p_{\mathrm{m},2}$ are* morph interval equivalent *if and only if*

$$\Delta\,\mathrm{m}\,(\Delta p_{\mathrm{m},1}) = \Delta\,\mathrm{m}\,(\Delta p_{\mathrm{m},2})$$

The fact that two morphetic pitch intervals are morph interval equivalent is denoted as follows:

$$\Delta p_{\mathrm{m},1} \equiv_{\Delta\,\mathrm{m}} \Delta p_{\mathrm{m},2}$$

Equivalence relations between frequency intervals

Definition 333 **($\Delta f_1 \equiv_{\Delta\,\mathrm{p_c}} \Delta f_2$)** *Two frequency intervals Δf_1 and Δf_2 are* chromatic pitch interval equivalent *if and only if*

$$\Delta\,\mathrm{p_c}\,(\Delta f_1) = \Delta\,\mathrm{p_c}\,(\Delta f_2)$$

The fact that two frequency intervals are chromatic pitch interval equivalent is denoted as follows:

$$\Delta f_1 \equiv_{\Delta\,\mathrm{p_c}} \Delta f_2$$

Definition 334 **($\Delta f_1 \equiv_{\Delta\,\mathrm{c}} \Delta f_2$)** *Two frequency intervals Δf_1 and Δf_2 are* chroma interval equivalent *if and only if*

$$\Delta\,\mathrm{c}\,(\Delta f_1) = \Delta\,\mathrm{c}\,(\Delta f_2)$$

The fact that two frequency intervals are chroma interval equivalent is denoted as follows:

$$\Delta f_1 \equiv_{\Delta\,\mathrm{c}} \Delta f_2$$

Equivalence relations between chromamorph intervals

Definition 335 **($\Delta q_1 \equiv_{\Delta\,\mathrm{c}} \Delta q_2$)** *Two chromamorph intervals Δq_1 and Δq_2 are* chroma interval equivalent *if and only if*

$$\Delta\,\mathrm{c}\,(\Delta q_1) = \Delta\,\mathrm{c}\,(\Delta q_2)$$

The fact that two chromamorph intervals are chroma interval equivalent is denoted as follows:

$$\Delta q_1 \equiv_{\Delta\,\mathrm{c}} \Delta q_2$$

Definition 336 **($\Delta q_1 \equiv_{\Delta\,\mathrm{m}} \Delta q_2$)** *Two chromamorph intervals Δq_1 and Δq_2 are* morph interval equivalent *if and only if*

$$\Delta\,\mathrm{m}\,(\Delta q_1) = \Delta\,\mathrm{m}\,(\Delta q_2)$$

The fact that two chromamorph intervals are morph interval equivalent is denoted as follows:

$$\Delta q_1 \equiv_{\Delta\,\mathrm{m}} \Delta q_2$$

Equivalence relations between chromatic genus intervals

Definition 337 **($\Delta g_{\mathrm{c},1} \equiv_{\Delta\,\mathrm{c}} \Delta g_{\mathrm{c},2}$)** *Two chromatic genus intervals $\Delta g_{\mathrm{c},1}$ and $\Delta g_{\mathrm{c},2}$ are* chroma interval equivalent *if and only if*

$$\Delta\,\mathrm{c}\,(\Delta g_{\mathrm{c},1}) = \Delta\,\mathrm{c}\,(\Delta g_{\mathrm{c},2})$$

The fact that two chromatic genus intervals are chroma interval equivalent is denoted as follows:

$$\Delta g_{\mathrm{c},1} \equiv_{\Delta\,\mathrm{c}} \Delta g_{\mathrm{c},2}$$

Equivalence relations between genus intervals

Definition 338 ($\Delta g_1 \equiv_{\Delta\,\mathrm{c}} \Delta g_2$) *Two genus intervals Δg_1 and Δg_2 are* chroma interval equivalent *if and only if*

$$\Delta\,\mathrm{c}\,(\Delta g_1) = \Delta\,\mathrm{c}\,(\Delta g_2)$$

The fact that two genus intervals are chroma interval equivalent is denoted as follows:

$$\Delta g_1 \equiv_{\Delta\,\mathrm{c}} \Delta g_2$$

Definition 339 ($\Delta g_1 \equiv_{\Delta\,\mathrm{m}} \Delta g_2$) *Two genus intervals Δg_1 and Δg_2 are* morph interval equivalent *if and only if*

$$\Delta\,\mathrm{m}\,(\Delta g_1) = \Delta\,\mathrm{m}\,(\Delta g_2)$$

The fact that two genus intervals are morph interval equivalent is denoted as follows:

$$\Delta g_1 \equiv_{\Delta\,\mathrm{m}} \Delta g_2$$

Theorem 340 *Morph interval equivalence of genus intervals is transitive. In other words, if Δg_1, Δg_2 and Δg_3 are any three genus intervals in a specified pitch system, then*

$$(\Delta g_1 \equiv_{\Delta\,\mathrm{m}} \Delta g_2) \wedge (\Delta g_2 \equiv_{\Delta\,\mathrm{m}} \Delta g_3) \Rightarrow (\Delta g_1 \equiv_{\Delta\,\mathrm{m}} \Delta g_3)$$

Proof

R1	Let		$\Delta g_1 \equiv_{\Delta\,\mathrm{m}} \Delta g_2$
R2	Let		$\Delta g_2 \equiv_{\Delta\,\mathrm{m}} \Delta g_3$
R3	R1 & 339	$\Rightarrow$	$\Delta\,\mathrm{m}\,(\Delta g_1) = \Delta\,\mathrm{m}\,(\Delta g_2)$
R4	R2 & 339	$\Rightarrow$	$\Delta\,\mathrm{m}\,(\Delta g_2) = \Delta\,\mathrm{m}\,(\Delta g_3)$
R5	R3 & R4	$\Rightarrow$	$\Delta\,\mathrm{m}\,(\Delta g_1) = \Delta\,\mathrm{m}\,(\Delta g_3)$
R6	R5 & 339	$\Rightarrow$	$\Delta g_1 \equiv_{\Delta\,\mathrm{m}} \Delta g_3$
R7	R1 to R6	$\Rightarrow$	$(\Delta g_1 \equiv_{\Delta\,\mathrm{m}} \Delta g_2) \wedge (\Delta g_2 \equiv_{\Delta\,\mathrm{m}} \Delta g_3) \Rightarrow (\Delta g_1 \equiv_{\Delta\,\mathrm{m}} \Delta g_3)$

Theorem 341 *Morph interval equivalence of genus intervals is symmetric. In other words, if Δg_1 and Δg_2 are any two genus intervals in a specified pitch system, then*

$$(\Delta g_1 \equiv_{\Delta\,\mathrm{m}} \Delta g_2) \iff (\Delta g_2 \equiv_{\Delta\,\mathrm{m}} \Delta g_1)$$

Proof

R1	Let		Δg_1 and Δg_2 be any two genus intervals in a pitch system.
R2	Let		$\Delta g_1 \equiv_{\Delta\,\mathrm{m}} \Delta g_2$
R3	R2 & 339	$\Rightarrow$	$\Delta\,\mathrm{m}\,(\Delta g_1) = \Delta\,\mathrm{m}\,(\Delta g_2)$
R4	R3 & 339	$\Rightarrow$	$\Delta g_2 \equiv_{\Delta\,\mathrm{m}} \Delta g_1$
R5	R1 to R4	$\Rightarrow$	$(\Delta g_1 \equiv_{\Delta\,\mathrm{m}} \Delta g_2) \Rightarrow (\Delta g_2 \equiv_{\Delta\,\mathrm{m}} \Delta g_1)$
R6	R5 & R1	$\Rightarrow$	$(\Delta g_2 \equiv_{\Delta\,\mathrm{m}} \Delta g_1) \Rightarrow (\Delta g_1 \equiv_{\Delta\,\mathrm{m}} \Delta g_2)$
R7	R5 & R6	$\Rightarrow$	$(\Delta g_1 \equiv_{\Delta\,\mathrm{m}} \Delta g_2) \iff (\Delta g_2 \equiv_{\Delta\,\mathrm{m}} \Delta g_1)$

Theorem 342 *Morph interval equivalence of genus intervals is reflexive. In other words, if Δg is any genus interval in a specified pitch system, then*

$$\Delta g \equiv_{\Delta\,\mathrm{m}} \Delta g$$

Proof

R1			$\Delta\,\mathrm{m}\,(\Delta g) = \Delta\,\mathrm{m}\,(\Delta g)$
R2	R1 & 339	$\Rightarrow$	$\Delta g \equiv_{\Delta\,\mathrm{m}} \Delta g$

Theorem 343 *Morph interval equivalence of genus intervals is an equivalence relation.*

Proof

R1	340	$\Rightarrow$	Morph interval equivalence of genus intervals is transitive.
R2	341	$\Rightarrow$	Morph interval equivalence of genus intervals is symmetric.
R3	342	$\Rightarrow$	Morph interval equivalence of genus intervals is reflexive.
R4	R1, R2	R3 & $\Rightarrow$	Morph interval equivalence of genus intervals is an equivalence relation.

Definition 344 ($\Delta g_1 \equiv_{\Delta\,\mathrm{g_c}} \Delta g_2$) *Two genus intervals Δg_1 and Δg_2 are* chromatic genus interval equivalent *if and only if*

$$\Delta\,\mathrm{g_c}\,(\Delta g_1) = \Delta\,\mathrm{g_c}\,(\Delta g_2)$$

The fact that two genus intervals are chromatic genus interval equivalent is denoted as follows:

$$\Delta g_1 \equiv_{\Delta\,\mathrm{g_c}} \Delta g_2$$

Definition 345 ($\Delta g_1 \equiv_{\Delta\,\mathrm{q}} \Delta g_2$) *Two genus intervals Δg_1 and Δg_2 are* chromamorph interval equivalent *if and only if*

$$\Delta\,\mathrm{q}\,(\Delta g_1) = \Delta\,\mathrm{q}\,(\Delta g_2)$$

The fact that two genus intervals are chromamorph interval equivalent is denoted as follows:

$$\Delta g_1 \equiv_{\Delta\,\mathrm{q}} \Delta g_2$$

4.4.4 Inequalities between *MIPS* intervals

Inequalities between two pitch intervals

Definition 346 *If Δp_1 and Δp_2 are any two pitch intervals in a pitch system ψ then Δp_1 is* chromatic pitch interval less than Δp_2*, denoted*

$$\Delta p_1 <_{\Delta\,\mathrm{p_c}} \Delta p_2$$

if and only if

$$\Delta\,\mathrm{p_c}\,(\Delta p_1) < \Delta\,\mathrm{p_c}\,(\Delta p_2)$$

Definition 347 *If Δp_1 and Δp_2 are any two pitch intervals in a pitch system ψ then Δp_1 is* chromatic pitch interval less than or equal to Δp_2*, denoted*

$$\Delta p_1 \leq_{\Delta\,\mathrm{p_c}} \Delta p_2$$

if and only if

$$\Delta\,\mathrm{p_c}\,(\Delta p_1) \leq \Delta\,\mathrm{p_c}\,(\Delta p_2)$$

Definition 348 *If Δp_1 and Δp_2 are any two pitch intervals in a pitch system ψ then Δp_1 is* chromatic pitch interval greater than Δp_2*, denoted*

$$\Delta p_1 >_{\Delta\,\mathrm{p_c}} \Delta p_2$$

if and only if

$$\Delta\,\mathrm{p_c}\,(\Delta p_1) > \Delta\,\mathrm{p_c}\,(\Delta p_2)$$

Definition 349 *If Δp_1 and Δp_2 are any two pitch intervals in a pitch system ψ then Δp_1 is* chromatic pitch interval greater than or equal to Δp_2*, denoted*

$$\Delta p_1 \geq_{\Delta\,\mathrm{p_c}} \Delta p_2$$

if and only if

$$\Delta\,\mathrm{p_c}\,(\Delta p_1) \geq \Delta\,\mathrm{p_c}\,(\Delta p_2)$$

Definition 350 *If Δp_1 and Δp_2 are any two pitch intervals in a pitch system ψ then Δp_1 is* morphetic pitch interval less than Δp_2*, denoted*

$$\Delta p_1 <_{\Delta\,\mathrm{p_m}} \Delta p_2$$

if and only if

$$\Delta\,\mathrm{p_m}\,(\Delta p_1) < \Delta\,\mathrm{p_m}\,(\Delta p_2)$$

Definition 351 *If Δp_1 and Δp_2 are any two pitch intervals in a pitch system ψ then Δp_1 is* morphetic pitch interval less than or equal to Δp_2*, denoted*

$$\Delta p_1 \leq_{\Delta\,\mathrm{p_m}} \Delta p_2$$

if and only if

$$\Delta\,\mathrm{p_m}\,(\Delta p_1) \leq \Delta\,\mathrm{p_m}\,(\Delta p_2)$$

Definition 352 *If Δp_1 and Δp_2 are any two pitch intervals in a pitch system ψ then Δp_1 is* morphetic pitch interval greater than Δp_2, *denoted*

$$\Delta p_1 >_{\Delta\,\mathrm{p_m}} \Delta p_2$$

if and only if

$$\Delta\,\mathrm{p_m}\,(\Delta p_1) > \Delta\,\mathrm{p_m}\,(\Delta p_2)$$

Definition 353 *If Δp_1 and Δp_2 are any two pitch intervals in a pitch system ψ then Δp_1 is* morphetic pitch interval greater than or equal to Δp_2, *denoted*

$$\Delta p_1 \geq_{\Delta\,\mathrm{p_m}} \Delta p_2$$

if and only if

$$\Delta\,\mathrm{p_m}\,(\Delta p_1) \geq \Delta\,\mathrm{p_m}\,(\Delta p_2)$$

Definition 354 *If Δp_1 and Δp_2 are any two pitch intervals in a pitch system ψ then Δp_1 is* frequency interval less than Δp_2, *denoted*

$$\Delta p_1 <_{\Delta\,\mathrm{f}} \Delta p_2$$

if and only if

$$\Delta\,\mathrm{f}\,(\Delta p_1) < \Delta\,\mathrm{f}\,(\Delta p_2)$$

Definition 355 *If Δp_1 and Δp_2 are any two pitch intervals in a pitch system ψ then Δp_1 is* frequency interval less than or equal to Δp_2, *denoted*

$$\Delta p_1 \leq_{\Delta\,\mathrm{f}} \Delta p_2$$

if and only if

$$\Delta\,\mathrm{f}\,(\Delta p_1) \leq \Delta\,\mathrm{f}\,(\Delta p_2)$$

Definition 356 *If Δp_1 and Δp_2 are any two pitch intervals in a pitch system ψ then Δp_1 is* frequency interval greater than Δp_2, *denoted*

$$\Delta p_1 >_{\Delta\,\mathrm{f}} \Delta p_2$$

if and only if

$$\Delta\,\mathrm{f}\,(\Delta p_1) > \Delta\,\mathrm{f}\,(\Delta p_2)$$

Definition 357 *If Δp_1 and Δp_2 are any two pitch intervals in a pitch system ψ then Δp_1 is* frequency interval greater than or equal to Δp_2, *denoted*

$$\Delta p_1 \geq_{\Delta\,\mathrm{f}} \Delta p_2$$

if and only if

$$\Delta\,\mathrm{f}\,(\Delta p_1) \geq \Delta\,\mathrm{f}\,(\Delta p_2)$$

Definition 358 *If Δp_1 and Δp_2 are any two pitch intervals in a pitch system ψ then Δp_1 is* chroma interval less than Δp_2, *denoted*

$$\Delta p_1 <_{\Delta\,\mathrm{c}} \Delta p_2$$

if and only if

$$\Delta\,\mathrm{c}\,(\Delta p_1) < \Delta\,\mathrm{c}\,(\Delta p_2)$$

Definition 359 *If* Δp_1 *and* Δp_2 *are any two pitch intervals in a pitch system* ψ *then* Δp_1 *is* chroma interval less than or equal to Δp_2, *denoted*

$$\Delta p_1 \leq_{\Delta \mathrm{c}} \Delta p_2$$

if and only if

$$\Delta \mathrm{c}\,(\Delta p_1) \leq \Delta \mathrm{c}\,(\Delta p_2)$$

Definition 360 *If* Δp_1 *and* Δp_2 *are any two pitch intervals in a pitch system* ψ *then* Δp_1 *is* chroma interval greater than Δp_2, *denoted*

$$\Delta p_1 >_{\Delta \mathrm{c}} \Delta p_2$$

if and only if

$$\Delta \mathrm{c}\,(\Delta p_1) > \Delta \mathrm{c}\,(\Delta p_2)$$

Definition 361 *If* Δp_1 *and* Δp_2 *are any two pitch intervals in a pitch system* ψ *then* Δp_1 *is* chroma interval greater than or equal to Δp_2, *denoted*

$$\Delta p_1 \geq_{\Delta \mathrm{c}} \Delta p_2$$

if and only if

$$\Delta \mathrm{c}\,(\Delta p_1) \geq \Delta \mathrm{c}\,(\Delta p_2)$$

Definition 362 *If* Δp_1 *and* Δp_2 *are any two pitch intervals in a pitch system* ψ *then* Δp_1 *is* morph interval less than Δp_2, *denoted*

$$\Delta p_1 <_{\Delta \mathrm{m}} \Delta p_2$$

if and only if

$$\Delta \mathrm{m}\,(\Delta p_1) < \Delta \mathrm{m}\,(\Delta p_2)$$

Definition 363 *If* Δp_1 *and* Δp_2 *are any two pitch intervals in a pitch system* ψ *then* Δp_1 *is* morph interval less than or equal to Δp_2, *denoted*

$$\Delta p_1 \leq_{\Delta \mathrm{m}} \Delta p_2$$

if and only if

$$\Delta \mathrm{m}\,(\Delta p_1) \leq \Delta \mathrm{m}\,(\Delta p_2)$$

Definition 364 *If* Δp_1 *and* Δp_2 *are any two pitch intervals in a pitch system* ψ *then* Δp_1 *is* morph interval greater than Δp_2, *denoted*

$$\Delta p_1 >_{\Delta \mathrm{m}} \Delta p_2$$

if and only if

$$\Delta \mathrm{m}\,(\Delta p_1) > \Delta \mathrm{m}\,(\Delta p_2)$$

Definition 365 *If* Δp_1 *and* Δp_2 *are any two pitch intervals in a pitch system* ψ *then* Δp_1 *is* morph interval greater than or equal to Δp_2, *denoted*

$$\Delta p_1 \geq_{\Delta \mathrm{m}} \Delta p_2$$

if and only if

$$\Delta \mathrm{m}\,(\Delta p_1) \geq \Delta \mathrm{m}\,(\Delta p_2)$$

Definition 366 *If* Δp_1 *and* Δp_2 *are any two pitch intervals in a pitch system* ψ *then* Δp_1 *is* chromatic genus interval less than Δp_2, *denoted*

$$\Delta p_1 <_{\Delta\,\mathrm{g_c}} \Delta p_2$$

if and only if

$$\Delta\,\mathrm{g_c}\,(\Delta p_1) < \Delta\,\mathrm{g_c}\,(\Delta p_2)$$

Definition 367 *If* Δp_1 *and* Δp_2 *are any two pitch intervals in a pitch system* ψ *then* Δp_1 *is* chromatic genus interval less than or equal to Δp_2, *denoted*

$$\Delta p_1 \leq_{\Delta\,\mathrm{g_c}} \Delta p_2$$

if and only if

$$\Delta\,\mathrm{g_c}\,(\Delta p_1) \leq \Delta\,\mathrm{g_c}\,(\Delta p_2)$$

Definition 368 *If* Δp_1 *and* Δp_2 *are any two pitch intervals in a pitch system* ψ *then* Δp_1 *is* chromatic genus interval greater than Δp_2, *denoted*

$$\Delta p_1 >_{\Delta\,\mathrm{g_c}} \Delta p_2$$

if and only if

$$\Delta\,\mathrm{g_c}\,(\Delta p_1) > \Delta\,\mathrm{g_c}\,(\Delta p_2)$$

Definition 369 *If* Δp_1 *and* Δp_2 *are any two pitch intervals in a pitch system* ψ *then* Δp_1 *is* chromatic genus interval greater than or equal to Δp_2, *denoted*

$$\Delta p_1 \geq_{\Delta\,\mathrm{g_c}} \Delta p_2$$

if and only if

$$\Delta\,\mathrm{g_c}\,(\Delta p_1) \geq \Delta\,\mathrm{g_c}\,(\Delta p_2)$$

Inequalities between two chromatic pitch intervals

Definition 370 *If* $\Delta p_{\mathrm{c},1}$ *and* $\Delta p_{\mathrm{c},2}$ *are any two chromatic pitch intervals in a pitch system* ψ *then* $\Delta p_{\mathrm{c},1}$ *is* chroma interval less than $\Delta p_{\mathrm{c},2}$, *denoted*

$$\Delta p_{\mathrm{c},1} <_{\Delta\,\mathrm{c}} \Delta p_{\mathrm{c},2}$$

if and only if

$$\Delta\,\mathrm{c}\,(\Delta p_{\mathrm{c},1}) < \Delta\,\mathrm{c}\,(\Delta p_{\mathrm{c},2})$$

Definition 371 *If* $\Delta p_{\mathrm{c},1}$ *and* $\Delta p_{\mathrm{c},2}$ *are any two chromatic pitch intervals in a pitch system* ψ *then* $\Delta p_{\mathrm{c},1}$ *is* chroma interval less than or equal to $\Delta p_{\mathrm{c},2}$, *denoted*

$$\Delta p_{\mathrm{c},1} \leq_{\Delta\,\mathrm{c}} \Delta p_{\mathrm{c},2}$$

if and only if

$$\Delta\,\mathrm{c}\,(\Delta p_{\mathrm{c},1}) \leq \Delta\,\mathrm{c}\,(\Delta p_{\mathrm{c},2})$$

Definition 372 *If* $\Delta p_{\mathrm{c},1}$ *and* $\Delta p_{\mathrm{c},2}$ *are any two chromatic pitch intervals in a pitch system* ψ *then* $\Delta p_{\mathrm{c},1}$ *is* chroma interval greater than $\Delta p_{\mathrm{c},2}$, *denoted*

$$\Delta p_{\mathrm{c},1} >_{\Delta\,\mathrm{c}} \Delta p_{\mathrm{c},2}$$

if and only if

$$\Delta\,\mathrm{c}\,(\Delta p_{\mathrm{c},1}) > \Delta\,\mathrm{c}\,(\Delta p_{\mathrm{c},2})$$

Definition 373 *If* $\Delta p_{\mathrm{c},1}$ *and* $\Delta p_{\mathrm{c},2}$ *are any two chromatic pitch intervals in a pitch system* ψ *then* $\Delta p_{\mathrm{c},1}$ *is* chroma interval greater than or equal to $\Delta p_{\mathrm{c},2}$*, denoted*

$$\Delta p_{\mathrm{c},1} \geq_{\Delta\,\mathrm{c}} \Delta p_{\mathrm{c},2}$$

if and only if

$$\Delta\,\mathrm{c}\,(\Delta p_{\mathrm{c},1}) \geq \Delta\,\mathrm{c}\,(\Delta p_{\mathrm{c},2})$$

Definition 374 *If* $\Delta p_{\mathrm{c},1}$ *and* $\Delta p_{\mathrm{c},2}$ *are any two chromatic pitch intervals in a pitch system* ψ *then* $\Delta p_{\mathrm{c},1}$ *is* frequency interval less than $\Delta p_{\mathrm{c},2}$*, denoted*

$$\Delta p_{\mathrm{c},1} <_{\Delta\,\mathrm{f}} \Delta p_{\mathrm{c},2}$$

if and only if

$$\Delta\,\mathrm{f}\,(\Delta p_{\mathrm{c},1}) < \Delta\,\mathrm{f}\,(\Delta p_{\mathrm{c},2})$$

Definition 375 *If* $\Delta p_{\mathrm{c},1}$ *and* $\Delta p_{\mathrm{c},2}$ *are any two chromatic pitch intervals in a pitch system* ψ *then* $\Delta p_{\mathrm{c},1}$ *is* frequency interval less than or equal to $\Delta p_{\mathrm{c},2}$*, denoted*

$$\Delta p_{\mathrm{c},1} \leq_{\Delta\,\mathrm{f}} \Delta p_{\mathrm{c},2}$$

if and only if

$$\Delta\,\mathrm{f}\,(\Delta p_{\mathrm{c},1}) \leq \Delta\,\mathrm{f}\,(\Delta p_{\mathrm{c},2})$$

Definition 376 *If* $\Delta p_{\mathrm{c},1}$ *and* $\Delta p_{\mathrm{c},2}$ *are any two chromatic pitch intervals in a pitch system* ψ *then* $\Delta p_{\mathrm{c},1}$ *is* frequency interval greater than $\Delta p_{\mathrm{c},2}$*, denoted*

$$\Delta p_{\mathrm{c},1} >_{\Delta\,\mathrm{f}} \Delta p_{\mathrm{c},2}$$

if and only if

$$\Delta\,\mathrm{f}\,(\Delta p_{\mathrm{c},1}) > \Delta\,\mathrm{f}\,(\Delta p_{\mathrm{c},2})$$

Definition 377 *If* $\Delta p_{\mathrm{c},1}$ *and* $\Delta p_{\mathrm{c},2}$ *are any two chromatic pitch intervals in a pitch system* ψ *then* $\Delta p_{\mathrm{c},1}$ *is* frequency interval greater than or equal to $\Delta p_{\mathrm{c},2}$*, denoted*

$$\Delta p_{\mathrm{c},1} \geq_{\Delta\,\mathrm{f}} \Delta p_{\mathrm{c},2}$$

if and only if

$$\Delta\,\mathrm{f}\,(\Delta p_{\mathrm{c},1}) \geq \Delta\,\mathrm{f}\,(\Delta p_{\mathrm{c},2})$$

Inequalities between two morphetic pitch intervals

Definition 378 *If* $\Delta p_{\mathrm{m},1}$ *and* $\Delta p_{\mathrm{m},2}$ *are any two morphetic pitch intervals in a pitch system* ψ *then* $\Delta p_{\mathrm{m},1}$ *is* morph interval less than $\Delta p_{\mathrm{m},2}$*, denoted*

$$\Delta p_{\mathrm{m},1} <_{\Delta\,\mathrm{m}} \Delta p_{\mathrm{m},2}$$

if and only if

$$\Delta\,\mathrm{m}\,(\Delta p_{\mathrm{m},1}) < \Delta\,\mathrm{m}\,(\Delta p_{\mathrm{m},2})$$

Definition 379 *If* $\Delta p_{\mathrm{m},1}$ *and* $\Delta p_{\mathrm{m},2}$ *are any two morphetic pitch intervals in a pitch system* ψ *then* $\Delta p_{\mathrm{m},1}$ *is* morph interval less than or equal to $\Delta p_{\mathrm{m},2}$*, denoted*

$$\Delta p_{\mathrm{m},1} \leq_{\Delta\,\mathrm{m}} \Delta p_{\mathrm{m},2}$$

if and only if

$$\Delta\,\mathrm{m}\left(\Delta p_{\mathrm{m},1}\right) \leq \Delta\,\mathrm{m}\left(\Delta p_{\mathrm{m},2}\right)$$

Definition 380 *If* $\Delta p_{\mathrm{m},1}$ *and* $\Delta p_{\mathrm{m},2}$ *are any two morphetic pitch intervals in a pitch system* ψ *then* $\Delta p_{\mathrm{m},1}$ *is* morph interval greater than $\Delta p_{\mathrm{m},2}$*, denoted*

$$\Delta p_{\mathrm{m},1} >_{\Delta\,\mathrm{m}} \Delta p_{\mathrm{m},2}$$

if and only if

$$\Delta\,\mathrm{m}\left(\Delta p_{\mathrm{m},1}\right) > \Delta\,\mathrm{m}\left(\Delta p_{\mathrm{m},2}\right)$$

Definition 381 *If* $\Delta p_{\mathrm{m},1}$ *and* $\Delta p_{\mathrm{m},2}$ *are any two morphetic pitch intervals in a pitch system* ψ *then* $\Delta p_{\mathrm{m},1}$ *is* morph interval greater than or equal to $\Delta p_{\mathrm{m},2}$*, denoted*

$$\Delta p_{\mathrm{m},1} \geq_{\Delta\,\mathrm{m}} \Delta p_{\mathrm{m},2}$$

if and only if

$$\Delta\,\mathrm{m}\left(\Delta p_{\mathrm{m},1}\right) \geq \Delta\,\mathrm{m}\left(\Delta p_{\mathrm{m},2}\right)$$

Inequalities between two frequency intervals

Definition 382 *If* Δf_1 *and* Δf_2 *are any two frequency intervals in a pitch system* ψ *then* Δf_1 *is* chromatic pitch interval less than Δf_2*, denoted*

$$\Delta f_1 <_{\Delta\,\mathrm{p_c}} \Delta f_2$$

if and only if

$$\Delta\,\mathrm{p_c}\left(\Delta f_1\right) < \Delta\,\mathrm{p_c}\left(\Delta f_2\right)$$

Definition 383 *If* Δf_1 *and* Δf_2 *are any two frequency intervals in a pitch system* ψ *then* Δf_1 *is* chromatic pitch interval less than or equal to Δf_2*, denoted*

$$\Delta f_1 \leq_{\Delta\,\mathrm{p_c}} \Delta f_2$$

if and only if

$$\Delta\,\mathrm{p_c}\left(\Delta f_1\right) \leq \Delta\,\mathrm{p_c}\left(\Delta f_2\right)$$

Definition 384 *If* Δf_1 *and* Δf_2 *are any two frequency intervals in a pitch system* ψ *then* Δf_1 *is* chromatic pitch interval greater than Δf_2*, denoted*

$$\Delta f_1 >_{\Delta\,\mathrm{p_c}} \Delta f_2$$

if and only if

$$\Delta\,\mathrm{p_c}\left(\Delta f_1\right) > \Delta\,\mathrm{p_c}\left(\Delta f_2\right)$$

Definition 385 *If* Δf_1 *and* Δf_2 *are any two frequency intervals in a pitch system* ψ *then* Δf_1 *is* chromatic pitch interval greater than or equal to Δf_2*, denoted*

$$\Delta f_1 \geq_{\Delta\,\mathrm{p_c}} \Delta f_2$$

if and only if

$$\Delta\,\mathrm{p_c}\left(\Delta f_1\right) \geq \Delta\,\mathrm{p_c}\left(\Delta f_2\right)$$

Definition 386 *If* Δf_1 *and* Δf_2 *are any two frequency intervals in a pitch system* ψ *then* Δf_1 *is* chroma interval less than Δf_2*, denoted*

$$\Delta f_1 <_{\Delta\,\mathrm{c}} \Delta f_2$$

if and only if

$$\Delta\,\mathrm{c}\,(\Delta f_1) < \Delta\,\mathrm{c}\,(\Delta f_2)$$

Definition 387 *If* Δf_1 *and* Δf_2 *are any two frequency intervals in a pitch system* ψ *then* Δf_1 *is* chroma interval less than or equal to Δf_2*, denoted*

$$\Delta f_1 \leq_{\Delta\,\mathrm{c}} \Delta f_2$$

if and only if

$$\Delta\,\mathrm{c}\,(\Delta f_1) \leq \Delta\,\mathrm{c}\,(\Delta f_2)$$

Definition 388 *If* Δf_1 *and* Δf_2 *are any two frequency intervals in a pitch system* ψ *then* Δf_1 *is* chroma interval greater than Δf_2*, denoted*

$$\Delta f_1 >_{\Delta\,\mathrm{c}} \Delta f_2$$

if and only if

$$\Delta\,\mathrm{c}\,(\Delta f_1) > \Delta\,\mathrm{c}\,(\Delta f_2)$$

Definition 389 *If* Δf_1 *and* Δf_2 *are any two frequency intervals in a pitch system* ψ *then* Δf_1 *is* chroma interval greater than or equal to Δf_2*, denoted*

$$\Delta f_1 \geq_{\Delta\,\mathrm{c}} \Delta f_2$$

if and only if

$$\Delta\,\mathrm{c}\,(\Delta f_1) \geq \Delta\,\mathrm{c}\,(\Delta f_2)$$

Inequalities between two chromatic genus intervals

Definition 390 *If* $\Delta g_{\mathrm{c},1}$ *and* $\Delta g_{\mathrm{c},2}$ *are any two chromatic genus intervals in a pitch system* ψ *then* $\Delta g_{\mathrm{c},1}$ *is* chroma interval less than $\Delta g_{\mathrm{c},2}$*, denoted*

$$\Delta g_{\mathrm{c},1} <_{\Delta\,\mathrm{c}} \Delta g_{\mathrm{c},2}$$

if and only if

$$\Delta\,\mathrm{c}\,(\Delta g_{\mathrm{c},1}) < \Delta\,\mathrm{c}\,(\Delta g_{\mathrm{c},2})$$

Definition 391 *If* $\Delta g_{\mathrm{c},1}$ *and* $\Delta g_{\mathrm{c},2}$ *are any two chromatic genus intervals in a pitch system* ψ *then* $\Delta g_{\mathrm{c},1}$ *is* chroma interval less than or equal to $\Delta g_{\mathrm{c},2}$*, denoted*

$$\Delta g_{\mathrm{c},1} \leq_{\Delta\,\mathrm{c}} \Delta g_{\mathrm{c},2}$$

if and only if

$$\Delta\,\mathrm{c}\,(\Delta g_{\mathrm{c},1}) \leq \Delta\,\mathrm{c}\,(\Delta g_{\mathrm{c},2})$$

Definition 392 *If* $\Delta g_{\mathrm{c},1}$ *and* $\Delta g_{\mathrm{c},2}$ *are any two chromatic genus intervals in a pitch system* ψ *then* $\Delta g_{\mathrm{c},1}$ *is* chroma interval greater than $\Delta g_{\mathrm{c},2}$*, denoted*

$$\Delta g_{\mathrm{c},1} >_{\Delta\,\mathrm{c}} \Delta g_{\mathrm{c},2}$$

if and only if

$$\Delta\,\mathrm{c}\,(\Delta g_{\mathrm{c},1}) > \Delta\,\mathrm{c}\,(\Delta g_{\mathrm{c},2})$$

Definition 393 *If* $\Delta g_{c,1}$ *and* $\Delta g_{c,2}$ *are any two chromatic genus intervals in a pitch system* ψ *then* $\Delta g_{c,1}$ *is* chroma interval greater than or equal to $\Delta g_{c,2}$, *denoted*

$$\Delta g_{c,1} \geq_{\Delta\,\mathrm{c}} \Delta g_{c,2}$$

if and only if

$$\Delta\,\mathrm{c}\,(\Delta g_{c,1}) \geq \Delta\,\mathrm{c}\,(\Delta g_{c,2})$$

Inequalities between two genus intervals

Definition 394 *If* Δg_1 *and* Δg_2 *are any two genus intervals in a pitch system* ψ *then* Δg_1 *is* chromatic genus interval less than Δg_2, *denoted*

$$\Delta g_1 <_{\Delta\,\mathrm{g_c}} \Delta g_2$$

if and only if

$$\Delta\,\mathrm{g_c}\,(\Delta g_1) < \Delta\,\mathrm{g_c}\,(\Delta g_2)$$

Definition 395 *If* Δg_1 *and* Δg_2 *are any two genus intervals in a pitch system* ψ *then* Δg_1 *is* chromatic genus interval less than or equal to Δg_2, *denoted*

$$\Delta g_1 \leq_{\Delta\,\mathrm{g_c}} \Delta g_2$$

if and only if

$$\Delta\,\mathrm{g_c}\,(\Delta g_1) \leq \Delta\,\mathrm{g_c}\,(\Delta g_2)$$

Definition 396 *If* Δg_1 *and* Δg_2 *are any two genus intervals in a pitch system* ψ *then* Δg_1 *is* chromatic genus interval greater than Δg_2, *denoted*

$$\Delta g_1 >_{\Delta\,\mathrm{g_c}} \Delta g_2$$

if and only if

$$\Delta\,\mathrm{g_c}\,(\Delta g_1) > \Delta\,\mathrm{g_c}\,(\Delta g_2)$$

Definition 397 *If* Δg_1 *and* Δg_2 *are any two genus intervals in a pitch system* ψ *then* Δg_1 *is* chromatic genus interval greater than or equal to Δg_2, *denoted*

$$\Delta g_1 \geq_{\Delta\,\mathrm{g_c}} \Delta g_2$$

if and only if

$$\Delta\,\mathrm{g_c}\,(\Delta g_1) \geq \Delta\,\mathrm{g_c}\,(\Delta g_2)$$

Definition 398 *If* Δg_1 *and* Δg_2 *are any two genus intervals in a pitch system* ψ *then* Δg_1 *is* morph interval less than Δg_2, *denoted*

$$\Delta g_1 <_{\Delta\,\mathrm{m}} \Delta g_2$$

if and only if

$$\Delta\,\mathrm{m}\,(\Delta g_1) < \Delta\,\mathrm{m}\,(\Delta g_2)$$

Definition 399 *If* Δg_1 *and* Δg_2 *are any two genus intervals in a pitch system* ψ *then* Δg_1 *is* morph interval less than or equal to Δg_2, *denoted*

$$\Delta g_1 \leq_{\Delta\,\mathrm{m}} \Delta g_2$$

if and only if

$$\Delta\,\mathrm{m}\,(\Delta g_1) \leq \Delta\,\mathrm{m}\,(\Delta g_2)$$

Definition 400 *If* Δg_1 *and* Δg_2 *are any two genus intervals in a pitch system* ψ *then* Δg_1 *is* morph interval greater than Δg_2, *denoted*

$$\Delta g_1 >_{\Delta\,\mathrm{m}} \Delta g_2$$

if and only if

$$\Delta\,\mathrm{m}\,(\Delta g_1) > \Delta\,\mathrm{m}\,(\Delta g_2)$$

Definition 401 *If* Δg_1 *and* Δg_2 *are any two genus intervals in a pitch system* ψ *then* Δg_1 *is* morph interval greater than or equal to Δg_2, *denoted*

$$\Delta g_1 \geq_{\Delta\,\mathrm{m}} \Delta g_2$$

if and only if

$$\Delta\,\mathrm{m}\,(\Delta g_1) \geq \Delta\,\mathrm{m}\,(\Delta g_2)$$

Definition 402 *If* Δg_1 *and* Δg_2 *are any two genus intervals in a pitch system* ψ *then* Δg_1 *is* chroma interval less than Δg_2, *denoted*

$$\Delta g_1 <_{\Delta\,\mathrm{c}} \Delta g_2$$

if and only if

$$\Delta\,\mathrm{c}\,(\Delta g_1) < \Delta\,\mathrm{c}\,(\Delta g_2)$$

Definition 403 *If* Δg_1 *and* Δg_2 *are any two genus intervals in a pitch system* ψ *then* Δg_1 *is* chroma interval less than or equal to Δg_2, *denoted*

$$\Delta g_1 \leq_{\Delta\,\mathrm{c}} \Delta g_2$$

if and only if

$$\Delta\,\mathrm{c}\,(\Delta g_1) \leq \Delta\,\mathrm{c}\,(\Delta g_2)$$

Definition 404 *If* Δg_1 *and* Δg_2 *are any two genus intervals in a pitch system* ψ *then* Δg_1 *is* chroma interval greater than Δg_2, *denoted*

$$\Delta g_1 >_{\Delta\,\mathrm{c}} \Delta g_2$$

if and only if

$$\Delta\,\mathrm{c}\,(\Delta g_1) > \Delta\,\mathrm{c}\,(\Delta g_2)$$

Definition 405 *If* Δg_1 *and* Δg_2 *are any two genus intervals in a pitch system* ψ *then* Δg_1 *is* chroma interval greater than or equal to Δg_2, *denoted*

$$\Delta g_1 \geq_{\Delta\,\mathrm{c}} \Delta g_2$$

if and only if

$$\Delta\,\mathrm{c}\,(\Delta g_1) \geq \Delta\,\mathrm{c}\,(\Delta g_2)$$

4.5 Transposing *MIPS* objects

4.5.1 Transposing a chroma

Definition 406 (Definition of $\tau_{\mathrm{c}}\,(c, \Delta c)$) *If* ψ *is a pitch system and* c_1 *and* c_2 *are chromae in* ψ *and* Δc *is a chroma interval in* ψ *then the* chroma transposition function *is defined as follows:*

$$\Delta\,\mathrm{c}\,(c_1, c_2) = \Delta c \Rightarrow \tau_{\mathrm{c}}\,(c_1, \Delta c) = c_2$$

Theorem 407 (Formula for $\tau_{\mathrm{c}}(c, \Delta c)$) *If c is a chroma and Δc is a chroma interval in a pitch system*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

then

$$\tau_{\mathrm{c}}(c, \Delta c) = (c + \Delta c) \bmod \mu_{\mathrm{c}}$$

Proof

R1	Let		$\Delta\,\mathrm{c}(c, c_2) = \Delta c$
R2	R1 & 406	$\Rightarrow$	$\tau_{\mathrm{c}}(c, \Delta c) = c_2$
R3	213	$\Rightarrow$	$\Delta\,\mathrm{c}(c, c_2) = (c_2 - c) \bmod \mu_{\mathrm{c}}$
R4	R1, R2 & R3	$\Rightarrow$	$\Delta c = (\tau_{\mathrm{c}}(c, \Delta c) - c) \bmod \mu_{\mathrm{c}}$
R5	214	$\Rightarrow$	$\mu_{\mathrm{c}} > \Delta c \geq 0$
R6	72 & 406	$\Rightarrow$	$\mu_{\mathrm{c}} > \tau_{\mathrm{c}}(c, \Delta c), c \geq 0$
R7	43, R4, R5 & R6	$\Rightarrow$	$\tau_{\mathrm{c}}(c, \Delta c) = (c + \Delta c) \bmod \mu_{\mathrm{c}}$

Theorem 408 *If ψ is a pitch system and c_1 and c_2 are chromae in ψ and Δc is a chroma interval in ψ then*

$$\tau_{\mathrm{c}}(c_1, \Delta c) = c_2 \Rightarrow \Delta\,\mathrm{c}(c_1, c_2) = \Delta c$$

Proof

R1	Let		$\tau_{\mathrm{c}}(c_1, \Delta c) = c_2$
R2	407	$\Rightarrow$	$\tau_{\mathrm{c}}(c_1, \Delta c) = (c_1 + \Delta c) \bmod \mu_{\mathrm{c}}$
R3	R1 & R2	$\Rightarrow$	$c_2 = (c_1 + \Delta c) \bmod \mu_{\mathrm{c}}$
R4	213	$\Rightarrow$	$\Delta\,\mathrm{c}(c_1, c_2) = (c_2 - c_1) \bmod \mu_{\mathrm{c}}$
R5	R3 & R4	$\Rightarrow$	$\Delta\,\mathrm{c}(c_1, c_2) = ((c_1 + \Delta c) \bmod \mu_{\mathrm{c}} - c_1) \bmod \mu_{\mathrm{c}}$
R6	R5 & 38	$\Rightarrow$	$\Delta\,\mathrm{c}(c_1, c_2) = (c_1 + \Delta c - c_1) \bmod \mu_{\mathrm{c}}$
			$= \Delta c \bmod \mu_{\mathrm{c}}$
R7	R6, 214 & 44	$\Rightarrow$	$\Delta\,\mathrm{c}(c_1, c_2) = \Delta c$
R8	R1 to R7	$\Rightarrow$	$\tau_{\mathrm{c}}(c_1, \Delta c) = c_2 \Rightarrow \Delta\,\mathrm{c}(c_1, c_2) = \Delta c$

Theorem 409 *If ψ is a pitch system and c_1 and c_2 are chromae in ψ and Δc is a chroma interval in ψ then*

$$\Delta\,\mathrm{c}\,(c_1, c_2) = \Delta c \iff \tau_{\mathrm{c}}\,(c_1, \Delta c) = c_2$$

Proof

R1	408	$\Rightarrow$	$\tau_{\mathrm{c}}\,(c_1, \Delta c) = c_2 \Rightarrow \Delta\,\mathrm{c}\,(c_1, c_2) = \Delta c$
R2	406	$\Rightarrow$	$\Delta\,\mathrm{c}\,(c_1, c_2) = \Delta c \Rightarrow \tau_{\mathrm{c}}\,(c_1, \Delta c) = c_2$
R3	R1 & R2	$\Rightarrow$	$\Delta\,\mathrm{c}\,(c_1, c_2) = \Delta c \iff \tau_{\mathrm{c}}\,(c_1, \Delta c) = c_2$

Theorem 410 *If ψ is a pitch system and Δc_1 and Δc_2 are chroma intervals in ψ and c is a chroma in ψ then*

$$(\tau_{\mathrm{c}}\,(c, \Delta c_1) = \tau_{\mathrm{c}}\,(c, \Delta c_2)) \Rightarrow (\Delta c_1 = \Delta c_2)$$

Proof

R1	407	$\Rightarrow$	$\tau_{\mathrm{c}}\,(c, \Delta c_1) = (c + \Delta c_1) \bmod \mu_{\mathrm{c}}$
R2	407	$\Rightarrow$	$\tau_{\mathrm{c}}\,(c, \Delta c_2) = (c + \Delta c_2) \bmod \mu_{\mathrm{c}}$
R3	Let		$\tau_{\mathrm{c}}\,(c, \Delta c_1) = \tau_{\mathrm{c}}\,(c, \Delta c_2)$
R4	R1, R2 & R3	$\Rightarrow$	$(c + \Delta c_1) \bmod \mu_{\mathrm{c}} = (c + \Delta c_2) \bmod \mu_{\mathrm{c}}$
R5	214	$\Rightarrow$	$(\Delta c_1 \in \mathbb{Z}) \wedge (0 \leq \Delta c_1 < \mu_{\mathrm{c}})$
R6	214	$\Rightarrow$	$(\Delta c_2 \in \mathbb{Z}) \wedge (0 \leq \Delta c_2 < \mu_{\mathrm{c}})$
R7	Let		$\frac{\Delta c_1 - \Delta c_2}{\mu_{\mathrm{c}}} = n$
R8	R4, R7 & 40	$\Rightarrow$	n is an integer
R9	R7	$\Rightarrow$	$\Delta c_1 = n \times \mu_{\mathrm{c}} + \Delta c_2$
R10	R5, R6, R8 & R9	$\Rightarrow$	$n = 0$
R11	R9 & R10	$\Rightarrow$	$\Delta c_1 = \Delta c_2$
R12	R1 to R11	$\Rightarrow$	$(\tau_{\mathrm{c}}\,(c, \Delta c_1) = \tau_{\mathrm{c}}\,(c, \Delta c_2)) \Rightarrow (\Delta c_1 = \Delta c_2)$

4.5.2 Transposing a morph

Definition 411 (Morph transposition function) *If ψ is a pitch system and m_1 and m_2 are morphs in ψ and Δm is a morph interval in ψ then the* morph transposition function *is defined as follows:*

$$\Delta\,\mathrm{m}\,(m_1, m_2) = \Delta m \Rightarrow \tau_{\mathrm{m}}\,(m_1, \Delta m) = m_2$$

Theorem 412 (Formula for morph transposition function) *If m is a morph and Δm is a morph interval in a pitch system*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

then

$$\tau_{\mathrm{m}}(m, \Delta m) = (m + \Delta m) \bmod \mu_{\mathrm{m}}$$

Proof

R1	Let		$\Delta\,\mathrm{m}(m, m_2) = \Delta m$
R2	R1 & 411	$\Rightarrow$	$\tau_{\mathrm{m}}(m, \Delta m) = m_2$
R3	217	$\Rightarrow$	$\Delta\,\mathrm{m}(m, m_2) = (m_2 - m) \bmod \mu_{\mathrm{m}}$
R4	R1, R2 & R3	$\Rightarrow$	$\Delta m = (\tau_{\mathrm{m}}(m, \Delta m) - m) \bmod \mu_{\mathrm{m}}$
R5	218	$\Rightarrow$	$\mu_{\mathrm{m}} > \Delta m \geq 0$
R6	77 & 411	$\Rightarrow$	$\mu_{\mathrm{m}} > \tau_{\mathrm{m}}(m, \Delta m), m \geq 0$
R7	43, R4, R5 & R6	$\Rightarrow$	$\tau_{\mathrm{m}}(m, \Delta m) = (m + \Delta m) \bmod \mu_{\mathrm{m}}$

Theorem 413 *If ψ is a pitch system and m_1 and m_2 are morphs in ψ and Δm is a morph interval in ψ then*

$$\tau_{\mathrm{m}}(m_1, \Delta m) = m_2 \Rightarrow \Delta\,\mathrm{m}(m_1, m_2) = \Delta m$$

Proof

R1	Let		$\tau_{\mathrm{m}}(m_1, \Delta m) = m_2$
R2	412	$\Rightarrow$	$\tau_{\mathrm{m}}(m_1, \Delta m) = (m_1 + \Delta m) \bmod \mu_{\mathrm{m}}$
R3	R1 & R2	$\Rightarrow$	$m_2 = (m_1 + \Delta m) \bmod \mu_{\mathrm{m}}$
R4	217	$\Rightarrow$	$\Delta\,\mathrm{m}(m_1, m_2) = (m_2 - m_1) \bmod \mu_{\mathrm{m}}$
R5	R3 & R4	$\Rightarrow$	$\Delta\,\mathrm{m}(m_1, m_2) = ((m_1 + \Delta m) \bmod \mu_{\mathrm{m}} - m_1) \bmod \mu_{\mathrm{m}}$
R6	R5 & 38	$\Rightarrow$	$\Delta\,\mathrm{m}(m_1, m_2) = (m_1 + \Delta m - m_1) \bmod \mu_{\mathrm{m}}$
			$= \Delta m \bmod \mu_{\mathrm{m}}$
R7	R6, 218 & 44	$\Rightarrow$	$\Delta\,\mathrm{m}(m_1, m_2) = \Delta m$
R8	R1 to R7	$\Rightarrow$	$\tau_{\mathrm{m}}(m_1, \Delta m) = m_2 \Rightarrow \Delta\,\mathrm{m}(m_1, m_2) = \Delta m$

Theorem 414 *If ψ is a pitch system and m_1 and m_2 are morphs in ψ and Δm is a morph interval in ψ then*

$$\Delta\,\mathrm{m}\,(m_1, m_2) = \Delta m \iff \tau_{\mathrm{m}}\,(m_1, \Delta m) = m_2$$

Proof

R1	413	$\Rightarrow$	$\tau_{\mathrm{m}}\,(m_1, \Delta m) = m_2 \Rightarrow \Delta\,\mathrm{m}\,(m_1, m_2) = \Delta m$
R2	411	$\Rightarrow$	$\Delta\,\mathrm{m}\,(m_1, m_2) = \Delta m \Rightarrow \tau_{\mathrm{m}}\,(m_1, \Delta m) = m_2$
R3	R1 & R2	$\Rightarrow$	$\Delta\,\mathrm{m}\,(m_1, m_2) = \Delta m \iff \tau_{\mathrm{m}}\,(m_1, \Delta m) = m_2$

Theorem 415 *If ψ is a pitch system and Δm_1 and Δm_2 are morph intervals in ψ and m is a morph in ψ then*

$$(\tau_{\mathrm{m}}\,(m, \Delta m_1) = \tau_{\mathrm{m}}\,(m, \Delta m_2)) \Rightarrow (\Delta m_1 = \Delta m_2)$$

Proof

R1	412	$\Rightarrow$	$\tau_{\mathrm{m}}\,(m, \Delta m_1) = (m + \Delta m_1) \bmod \mu_{\mathrm{m}}$
R2	412	$\Rightarrow$	$\tau_{\mathrm{m}}\,(m, \Delta m_2) = (m + \Delta m_2) \bmod \mu_{\mathrm{m}}$
R3	Let		$\tau_{\mathrm{m}}\,(m, \Delta m_1) = \tau_{\mathrm{m}}\,(m, \Delta m_2)$
R4	R1, R2 & R3	$\Rightarrow$	$(m + \Delta m_1) \bmod \mu_{\mathrm{m}} = (m + \Delta m_2) \bmod \mu_{\mathrm{m}}$
R5	218	$\Rightarrow$	$(\Delta m_1 \in \mathbb{Z}) \wedge (0 \leq \Delta m_1 < \mu_{\mathrm{m}})$
R6	218	$\Rightarrow$	$(\Delta m_2 \in \mathbb{Z}) \wedge (0 \leq \Delta m_2 < \mu_{\mathrm{m}})$
R7	Let		$\frac{\Delta m_1 - \Delta m_2}{\mu_{\mathrm{m}}} = n$
R8	R4, R7 & 40	$\Rightarrow$	n is an integer
R9	R7	$\Rightarrow$	$\Delta m_1 = n \times \mu_{\mathrm{m}} + \Delta m_2$
R10	R5, R6, R8 & R9	$\Rightarrow$	$n = 0$
R11	R9 & R10	$\Rightarrow$	$\Delta m_1 = \Delta m_2$
R12	R1 to R11	$\Rightarrow$	$(\tau_{\mathrm{m}}\,(m, \Delta m_1) = \tau_{\mathrm{m}}\,(m, \Delta m_2)) \Rightarrow (\Delta m_1 = \Delta m_2)$

4.5.3 Transposing a chromamorph

Definition 416 (Definition of $\tau_{\mathrm{q}}\,(q, \Delta q)$) *If ψ is a pitch system and q_1 and q_2 are chromamorphs in ψ and Δq is a chromamorph interval in ψ then the* chromamorph transposition function *is defined as follows:*

$$\Delta\,\mathrm{q}\,(q_1, q_2) = \Delta q \Rightarrow \tau_{\mathrm{q}}\,(q_1, \Delta q) = q_2$$

Theorem 417 (Formula for $\tau_{\mathrm{q}}(q, \Delta q)$) *If q is a chromamorph and Δq is a chromamorph interval in a pitch system ψ then*

$$\tau_{\mathrm{q}}(q, \Delta q) = [\tau_{\mathrm{c}}(\mathrm{c}(q), \Delta\,\mathrm{c}(\Delta q)), \tau_{\mathrm{m}}(\mathrm{m}(q), \Delta\,\mathrm{m}(\Delta q))]$$

Proof

R1	Let		$\Delta\,\mathrm{q}\,(q, q_2) = \Delta q$
R2	416	$\Rightarrow$	$\tau_{\mathrm{q}}\,(q, \Delta q) = q_2$
R3	223	$\Rightarrow$	$\Delta\,\mathrm{q}\,(q, q_2) = [\Delta\,\mathrm{c}\,(q, q_2)\,, \Delta\,\mathrm{m}\,(q, q_2)]$
R4	221	$\Rightarrow$	$\Delta\,\mathrm{c}\,(q, q_2) = \Delta\,\mathrm{c}\,(\mathrm{c}\,(q)\,, \mathrm{c}\,(q_2))$
R5	222	$\Rightarrow$	$\Delta\,\mathrm{m}\,(q, q_2) = \Delta\,\mathrm{m}\,(\mathrm{m}\,(q)\,, \mathrm{m}\,(q_2))$
R6	213	$\Rightarrow$	$\Delta\,\mathrm{c}\,(\mathrm{c}\,(q)\,, \mathrm{c}\,(q_2)) = (\mathrm{c}\,(q_2) - \mathrm{c}\,(q)) \bmod \mu_{\mathrm{c}}$
R7	217	$\Rightarrow$	$\Delta\,\mathrm{m}\,(\mathrm{m}\,(q)\,, \mathrm{m}\,(q_2)) = (\mathrm{m}\,(q_2) - \mathrm{m}\,(q)) \bmod \mu_{\mathrm{m}}$
R8	R1 & 299	$\Rightarrow$	$\Delta\,\mathrm{c}\,(\Delta q) = \Delta\,\mathrm{c}\,(q, q_2)$
R9	R4, R6 & R8	$\Rightarrow$	$\Delta\,\mathrm{c}\,(\Delta q) = (\mathrm{c}\,(q_2) - \mathrm{c}\,(q)) \bmod \mu_{\mathrm{c}}$
R10	R1 & 302	$\Rightarrow$	$\Delta\,\mathrm{m}\,(\Delta q) = \Delta\,\mathrm{m}\,(q, q_2)$
R11	R5, R7 & R10	$\Rightarrow$	$\Delta\,\mathrm{m}\,(\Delta q) = (\mathrm{m}\,(q_2) - \mathrm{m}\,(q)) \bmod \mu_{\mathrm{m}}$
R12	72	$\Rightarrow$	$\mu_{\mathrm{c}} > \mathrm{c}\,(q)\,, \mathrm{c}\,(q_2) \geq 0$
R13	214	$\Rightarrow$	$\mu_{\mathrm{c}} > \Delta\,\mathrm{c}\,(\Delta q) \geq 0$
R14	R9, R12, R13 & 43	$\Rightarrow$	$\mathrm{c}\,(q_2) = (\mathrm{c}\,(q) + \Delta\,\mathrm{c}\,(\Delta q)) \bmod \mu_{\mathrm{c}}$
R15	77	$\Rightarrow$	$\mu_{\mathrm{m}} > \mathrm{m}\,(q)\,, \mathrm{m}\,(q_2) \geq 0$
R16	218	$\Rightarrow$	$\mu_{\mathrm{m}} > \Delta\,\mathrm{m}\,(\Delta q) \geq 0$
R17	R11, R15, R16 & 43	$\Rightarrow$	$\mathrm{m}\,(q_2) = (\mathrm{m}\,(q) + \Delta\,\mathrm{m}\,(\Delta q)) \bmod \mu_{\mathrm{m}}$
R18	R14 & 407	$\Rightarrow$	$\tau_{\mathrm{c}}\,(\mathrm{c}\,(q)\,, \Delta\,\mathrm{c}\,(\Delta q)) = \mathrm{c}\,(q_2)$
R19	R17 & 412	$\Rightarrow$	$\tau_{\mathrm{m}}\,(\mathrm{m}\,(q)\,, \Delta\,\mathrm{m}\,(\Delta q)) = \mathrm{m}\,(q_2)$
R20	Let		$q_2 = [c_2, m_2]$
R21	R20 & 106	$\Rightarrow$	$\mathrm{c}\,(q_2) = c_2$
R22	R20 & 108	$\Rightarrow$	$\mathrm{m}\,(q_2) = m_2$
R23	R20, R21 & R22	$\Rightarrow$	$q_2 = [\mathrm{c}\,(q_2)\,, \mathrm{m}\,(q_2)]$
R24	R2, R18 & R19	$\Rightarrow$	$\tau_{\mathrm{q}}\,(q, \Delta q) = [\tau_{\mathrm{c}}\,(\mathrm{c}\,(q)\,, \Delta\,\mathrm{c}\,(\Delta q))\,, \tau_{\mathrm{m}}\,(\mathrm{m}\,(q)\,, \Delta\,\mathrm{m}\,(\Delta q))]$

Theorem 418 *If ψ is a pitch system and q_1 and q_2 are chromamorphs in ψ and Δq is a chromamorph interval in ψ then*

$$\tau_{\mathrm{q}}(q_1, \Delta q) = q_2 \Rightarrow \Delta\,\mathrm{q}(q_1, q_2) = \Delta q$$

Proof

R1	Let		$\tau_{\mathrm{q}}(q_1, \Delta q) = q_2$
R2	417	$\Rightarrow$	$\tau_{\mathrm{q}}(q_1, \Delta q) = [\tau_{\mathrm{c}}(\mathrm{c}(q_1), \Delta\,\mathrm{c}(\Delta q)), \tau_{\mathrm{m}}(\mathrm{m}(q_1), \Delta\,\mathrm{m}(\Delta q))]$
R3	223	$\Rightarrow$	$\Delta\,\mathrm{q}(q_1, q_2) = [\Delta\,\mathrm{c}(q_1, q_2), \Delta\,\mathrm{m}(q_1, q_2)]$
R4	221	$\Rightarrow$	$\Delta\,\mathrm{c}(q_1, q_2) = \Delta\,\mathrm{c}(\mathrm{c}(q_1), \mathrm{c}(q_2))$
R5	222	$\Rightarrow$	$\Delta\,\mathrm{m}(q_1, q_2) = \Delta\,\mathrm{m}(\mathrm{m}(q_1), \mathrm{m}(q_2))$
R6	R3, R4 & R5	$\Rightarrow$	$\Delta\,\mathrm{q}(q_1, q_2) = [\Delta\,\mathrm{c}(\mathrm{c}(q_1), \mathrm{c}(q_2)), \Delta\,\mathrm{m}(\mathrm{m}(q_1), \mathrm{m}(q_2))]$
R7	109	$\Rightarrow$	$q_2 = [\mathrm{c}(q_2), \mathrm{m}(q_2)]$
R8	R1, R2 & R7	$\Rightarrow$	$\tau_{\mathrm{c}}(\mathrm{c}(q_1), \Delta\,\mathrm{c}(\Delta q)) = \mathrm{c}(q_2)$
R9	R1, R2 & R7	$\Rightarrow$	$\tau_{\mathrm{m}}(\mathrm{m}(q_1), \Delta\,\mathrm{m}(\Delta q)) = \mathrm{m}(q_2)$
R10	R8 & 408	$\Rightarrow$	$\Delta\,\mathrm{c}(\mathrm{c}(q_1), \mathrm{c}(q_2)) = \Delta\,\mathrm{c}(\Delta q)$
R11	R9 & 413	$\Rightarrow$	$\Delta\,\mathrm{m}(\mathrm{m}(q_1), \mathrm{m}(q_2)) = \Delta\,\mathrm{m}(\Delta q)$
R12	R6, R10 & R11	$\Rightarrow$	$\Delta\,\mathrm{q}(q_1, q_2) = [\Delta\,\mathrm{c}(\Delta q), \Delta\,\mathrm{m}(\Delta q)]$
R13	R12 & 305	$\Rightarrow$	$\Delta\,\mathrm{q}(q_1, q_2) = \Delta q$
R14	R1 to R13	$\Rightarrow$	$\tau_{\mathrm{q}}(q_1, \Delta q) = q_2 \Rightarrow \Delta\,\mathrm{q}(q_1, q_2) = \Delta q$

Theorem 419 *If ψ is a pitch system and q_1 and q_2 are chromamorphs in ψ and Δq is a chromamorph interval in ψ then*

$$\tau_{\mathrm{q}}(q_1, \Delta q) = q_2 \iff \Delta\,\mathrm{q}(q_1, q_2) = \Delta q$$

Proof

R1	418	$\Rightarrow$	$\tau_{\mathrm{q}}(q_1, \Delta q) = q_2 \Rightarrow \Delta\,\mathrm{q}(q_1, q_2) = \Delta q$
R2	416	$\Rightarrow$	$\Delta\,\mathrm{q}(q_1, q_2) = \Delta q \Rightarrow \tau_{\mathrm{q}}(q_1, \Delta q) = q_2$
R3	R1 & R2	$\Rightarrow$	$\Delta\,\mathrm{q}(q_1, q_2) = \Delta q \iff \tau_{\mathrm{q}}(q_1, \Delta q) = q_2$

Theorem 420 *If ψ is a pitch system and Δq_1 and Δq_2 are chromamorph intervals in ψ and q is a chromamorph in ψ then*

$$(\tau_{\mathrm{q}}(q, \Delta q_1) = \tau_{\mathrm{q}}(q, \Delta q_2)) \Rightarrow (\Delta q_1 = \Delta q_2)$$

Proof

R1	Let		$\tau_{\mathrm{q}}(q, \Delta q_1) = q_1$
R2	Let		$\tau_{\mathrm{q}}(q, \Delta q_2) = q_2$
R3	R1 & 417	$\Rightarrow$	$q_1 = [\tau_{\mathrm{c}}(\mathrm{c}(q), \Delta\,\mathrm{c}(\Delta q_1)), \tau_{\mathrm{m}}(\mathrm{m}(q), \Delta\,\mathrm{m}(\Delta q_1))]$
R4	R2 & 417	$\Rightarrow$	$q_1 = [\tau_{\mathrm{c}}(\mathrm{c}(q), \Delta\,\mathrm{c}(\Delta q_2)), \tau_{\mathrm{m}}(\mathrm{m}(q), \Delta\,\mathrm{m}(\Delta q_2))]$
R5	Let		$\tau_{\mathrm{q}}(q, \Delta q_1) = \tau_{\mathrm{q}}(q, \Delta q_2)$
R6	R1, R2 & R5	$\Rightarrow$	$q_1 = q_2$
R7	R3, R4 & R6	$\Rightarrow$	$\tau_{\mathrm{c}}(\mathrm{c}(q), \Delta\,\mathrm{c}(\Delta q_1)) = \tau_{\mathrm{c}}(\mathrm{c}(q), \Delta\,\mathrm{c}(\Delta q_2))$
R8	R3, R4 & R6	$\Rightarrow$	$\tau_{\mathrm{m}}(\mathrm{m}(q), \Delta\,\mathrm{m}(\Delta q_1)) = \tau_{\mathrm{m}}(\mathrm{m}(q), \Delta\,\mathrm{m}(\Delta q_2))$
R9	R7 & 410	$\Rightarrow$	$\Delta\,\mathrm{c}(\Delta q_1) = \Delta\,\mathrm{c}(\Delta q_2)$
R10	R8 & 415	$\Rightarrow$	$\Delta\,\mathrm{m}(\Delta q_1) = \Delta\,\mathrm{m}(\Delta q_2)$
R11	305	$\Rightarrow$	$\Delta q_1 = [\Delta\,\mathrm{c}(\Delta q_1), \Delta\,\mathrm{m}(\Delta q_1)]$
R12	305	$\Rightarrow$	$\Delta q_2 = [\Delta\,\mathrm{c}(\Delta q_2), \Delta\,\mathrm{m}(\Delta q_2)]$
R13	R9, R10, R11 & R12	$\Rightarrow$	$\Delta q_1 = \Delta q_2$
R14	R1 to R13	$\Rightarrow$	$(\tau_{\mathrm{q}}(q, \Delta q_1) = \tau_{\mathrm{q}}(q, \Delta q_2)) \Rightarrow (\Delta q_1 = \Delta q_2)$

4.5.4 Transposing a genus

Definition 421 (Genus transposition function) *If ψ is a pitch system and g_1 and g_2 are genera in ψ and Δg is a genus interval in ψ then the* genus transposition function *is defined as follows:*

$$\Delta\,\mathrm{g}(g_1, g_2) = \Delta g \Rightarrow \tau_{\mathrm{g}}(g_1, \Delta g) = g_2$$

Theorem 422 (Formula for genus transposition function) *If*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

is a pitch system and g is a genus in ψ and Δg is a genus interval in ψ then

$$\tau_{\mathrm{g}}(g, \Delta g) = [\mathrm{g_c}(g) + \Delta\,\mathrm{g_c}(\Delta g) - \mu_{\mathrm{c}} \times ((\mathrm{m}(g) + \Delta\,\mathrm{m}(\Delta g))\ \mathrm{div}\ \mu_{\mathrm{m}}), \tau_{\mathrm{m}}(\mathrm{m}(g), \Delta\,\mathrm{m}(\Delta g))]$$

Proof

R1	Let		$\Delta g = \Delta\,\mathrm{g}\,(g, g_2)$
R2	421 & R1	$\Rightarrow$	$\tau_{\mathrm{g}}\,(g, \Delta g) = g_2$
R3	231	$\Rightarrow$	$\Delta\,\mathrm{g}\,(g, g_2) = [\Delta\,\mathrm{g_c}\,(g, g_2)\,, \Delta\,\mathrm{m}\,(g, g_2)]$
R4	230	$\Rightarrow$	$\Delta\,\mathrm{g_c}\,(g, g_2) = \mathrm{g_c}\,(g_2) - \mathrm{g_c}\,(g) - \mu_{\mathrm{c}} \times ((\mathrm{m}\,(g_2) - \mathrm{m}\,(g))\ \mathrm{div}\ \mu_{\mathrm{m}})$
R5	228	$\Rightarrow$	$\Delta\,\mathrm{m}\,(g, g_2) = \Delta\,\mathrm{m}\,(\mathrm{m}\,(g)\,, \mathrm{m}\,(g_2))$
R6	R1 & 309	$\Rightarrow$	$\Delta\,\mathrm{g_c}\,(\Delta g) = \Delta\,\mathrm{g_c}\,(g, g_2)$
R7	R4 & R6	$\Rightarrow$	$\Delta\,\mathrm{g_c}\,(\Delta g) = \mathrm{g_c}\,(g_2) - \mathrm{g_c}\,(g) - \mu_{\mathrm{c}} \times ((\mathrm{m}\,(g_2) - \mathrm{m}\,(g))\ \mathrm{div}\ \mu_{\mathrm{m}})$
R8	315 & R1	$\Rightarrow$	$\Delta\,\mathrm{m}\,(\Delta g) = \Delta\,\mathrm{m}\,(g, g_2)$
R9	R5 & R8	$\Rightarrow$	$\Delta\,\mathrm{m}\,(\Delta g) = \Delta\,\mathrm{m}\,(\mathrm{m}\,(g)\,, \mathrm{m}\,(g_2))$
R10	R9 & 217	$\Rightarrow$	$\Delta\,\mathrm{m}\,(\Delta g) = (\mathrm{m}\,(g_2) - \mathrm{m}\,(g))\ \mathrm{mod}\ \mu_{\mathrm{m}}$
R11	R10, 43, 77 & 218	$\Rightarrow$	$\mathrm{m}\,(g_2) = (\mathrm{m}\,(g) + \Delta\,\mathrm{m}\,(\Delta g))\ \mathrm{mod}\ \mu_{\mathrm{m}}$
R12	R7 & R11	$\Rightarrow$	$\mathrm{g_c}\,(g_2) = \Delta\,\mathrm{g_c}\,(\Delta g) + \mathrm{g_c}\,(g)$ $+\mu_{\mathrm{c}} \times (((\mathrm{m}\,(g) + \Delta\,\mathrm{m}\,(\Delta g))\ \mathrm{mod}\ \mu_{\mathrm{m}} - \mathrm{m}\,(g))\ \mathrm{div}\ \mu_{\mathrm{m}})$
R13	R12 & 51	$\Rightarrow$	$((\mathrm{m}\,(g) + \Delta\,\mathrm{m}\,(\Delta g))\ \mathrm{mod}\ \mu_{\mathrm{m}} - \mathrm{m}\,(g))\ \mathrm{div}\ \mu_{\mathrm{m}}$ $= \mathrm{int}\left(\frac{\Delta\,\mathrm{m}(\Delta g)}{\mu_{\mathrm{m}}}\right) - ((\mathrm{m}\,(g) + \Delta\,\mathrm{m}\,(\Delta g))\ \mathrm{div}\ \mu_{\mathrm{m}})$
R14	218	$\Rightarrow$	$\mathrm{int}\left(\frac{\Delta\,\mathrm{m}(\Delta g)}{\mu_{\mathrm{m}}}\right) = 0$
R15	R13 & R14	$\Rightarrow$	$((\mathrm{m}\,(g) + \Delta\,\mathrm{m}\,(\Delta g))\ \mathrm{mod}\ \mu_{\mathrm{m}} - \mathrm{m}\,(g))\ \mathrm{div}\ \mu_{\mathrm{m}}$ $= -\left((\mathrm{m}\,(g) + \Delta\,\mathrm{m}\,(\Delta g))\ \mathrm{div}\ \mu_{\mathrm{m}}\right)$
R16	R12 & R15	$\Rightarrow$	$\mathrm{g_c}\,(g_2) = \mathrm{g_c}\,(g) + \Delta\,\mathrm{g_c}\,(\Delta g) - \mu_{\mathrm{c}} \times ((\mathrm{m}\,(g) + \Delta\,\mathrm{m}\,(\Delta g))\ \mathrm{div}\ \mu_{\mathrm{m}})$
R17	R11 & 412	$\Rightarrow$	$\mathrm{m}\,(g_2) = \tau_{\mathrm{m}}\,(\mathrm{m}\,(g)\,, \Delta\,\mathrm{m}\,(\Delta g))$
R18	R2, R16, R17 & 118	$\Rightarrow$	$\tau_{\mathrm{g}}\,(g, \Delta g) = \left[\begin{array}{l} \mathrm{g_c}\,(g) + \Delta\,\mathrm{g_c}\,(\Delta g) \\ -\mu_{\mathrm{c}} \times ((\mathrm{m}\,(g) + \Delta\,\mathrm{m}\,(\Delta g))\ \mathrm{div}\ \mu_{\mathrm{m}})\,, \\ \tau_{\mathrm{m}}\,(\mathrm{m}\,(g)\,, \Delta\,\mathrm{m}\,(\Delta g)) \end{array}\right]$

Theorem 423 *If ψ is a pitch system and g_1 and g_2 are genera in ψ and Δg is a genus interval in ψ then*

$$\tau_{\mathrm{g}}(g_1, \Delta g) = g_2 \Rightarrow \Delta\,\mathrm{g}(g_1, g_2) = \Delta g$$

Proof

R1	Let		$\tau_{\mathrm{g}}(g_1, \Delta g) = g_2$
R2	R1 & 422	$\Rightarrow$	$g_2 = \begin{bmatrix} \mathrm{g_c}(g_1) + \Delta\,\mathrm{g_c}(\Delta g) - \mu_{\mathrm{c}} \times ((\mathrm{m}(g_1) + \Delta\,\mathrm{m}(\Delta g))\ \mathrm{div}\ \mu_{\mathrm{m}}), \\ \tau_{\mathrm{m}}(\mathrm{m}(g_1), \Delta\,\mathrm{m}(\Delta g)) \end{bmatrix}$
R3	231	$\Rightarrow$	$\Delta\,\mathrm{g}(g_1, g_2) = [\Delta\,\mathrm{g_c}(g_1, g_2), \Delta\,\mathrm{m}(g_1, g_2)]$
R4	230	$\Rightarrow$	$\Delta\,\mathrm{g_c}(g_1, g_2) = \mathrm{g_c}(g_2) - \mathrm{g_c}(g_1) - \mu_{\mathrm{c}} \times ((\mathrm{m}(g_2) - \mathrm{m}(g_1))\ \mathrm{div}\ \mu_{\mathrm{m}})$
R5	R2 & 115	$\Rightarrow$	$\mathrm{g_c}(g_2) = \mathrm{g_c}(g_1) + \Delta\,\mathrm{g_c}(\Delta g) - \mu_{\mathrm{c}} \times ((\mathrm{m}(g_1) + \Delta\,\mathrm{m}(\Delta g))\ \mathrm{div}\ \mu_{\mathrm{m}})$
R6	R4 & R5	$\Rightarrow$	$\Delta\,\mathrm{g_c}(g_1, g_2) = \mathrm{g_c}(g_1) + \Delta\,\mathrm{g_c}(\Delta g) - \mu_{\mathrm{c}} \times ((\mathrm{m}(g_1) + \Delta\,\mathrm{m}(\Delta g))\ \mathrm{div}\ \mu_{\mathrm{m}})$
			$\quad - \mathrm{g_c}(g_1) - \mu_{\mathrm{c}} \times ((\mathrm{m}(g_2) - \mathrm{m}(g_1))\ \mathrm{div}\ \mu_{\mathrm{m}})$
			$= \Delta\,\mathrm{g_c}(\Delta g) - \mu_{\mathrm{c}} \times ((\mathrm{m}(g) + \Delta\,\mathrm{m}(\Delta g))\ \mathrm{div}\ \mu_{\mathrm{m}} + (\mathrm{m}(g_2) - \mathrm{m}(g_1))\ \mathrm{div}\ \mu_{\mathrm{m}})$
R7	R2 & 117	$\Rightarrow$	$\mathrm{m}(g_2) = \tau_{\mathrm{m}}(\mathrm{m}(g_1), \Delta\,\mathrm{m}(\Delta g))$
R8	R7 & 412	$\Rightarrow$	$\mathrm{m}(g_2) = (\mathrm{m}(g_1) + \Delta\,\mathrm{m}(\Delta g))\ \mathrm{mod}\ \mu_{\mathrm{m}}$
R9	R8	$\Rightarrow$	$(\mathrm{m}(g_2) - \mathrm{m}(g_1))\ \mathrm{div}\ \mu_{\mathrm{m}} = ((\mathrm{m}(g_1) + \Delta\,\mathrm{m}(\Delta g))\ \mathrm{mod}\ \mu_{\mathrm{m}} - \mathrm{m}(g_1))\ \mathrm{div}\ \mu_{\mathrm{m}}$
R10	R9 & 51	$\Rightarrow$	$(\mathrm{m}(g_2) - \mathrm{m}(g_1))\ \mathrm{div}\ \mu_{\mathrm{m}} = \mathrm{int}\left(\frac{\Delta\,\mathrm{m}(\Delta g)}{\mu_{\mathrm{m}}}\right) - ((\mathrm{m}(g_1) + \Delta\,\mathrm{m}(\Delta g))\ \mathrm{div}\ \mu_{\mathrm{m}})$
R11	218	$\Rightarrow$	$\mathrm{int}\left(\frac{\Delta\,\mathrm{m}(\Delta g)}{\mu_{\mathrm{m}}}\right) = 0$
R12	R10 & R11	$\Rightarrow$	$(\mathrm{m}(g_2) - \mathrm{m}(g_1))\ \mathrm{div}\ \mu_{\mathrm{m}} = -((\mathrm{m}(g_1) + \Delta\,\mathrm{m}(\Delta g))\ \mathrm{div}\ \mu_{\mathrm{m}})$
R13	R6 & R12	$\Rightarrow$	$\Delta\,\mathrm{g_c}(g_1, g_2) = \Delta\,\mathrm{g_c}(\Delta g) - \mu_{\mathrm{c}} \times \begin{pmatrix} (\mathrm{m}(g) + \Delta\,\mathrm{m}(\Delta g))\ \mathrm{div}\ \mu_{\mathrm{m}} \\ -((\mathrm{m}(g_1) + \Delta\,\mathrm{m}(\Delta g))\ \mathrm{div}\ \mu_{\mathrm{m}}) \end{pmatrix}$
			$= \Delta\,\mathrm{g_c}(\Delta g)$
R14	228	$\Rightarrow$	$\Delta\,\mathrm{m}(g_1, g_2) = \Delta\,\mathrm{m}(\mathrm{m}(g_1), \mathrm{m}(g_2))$
R15	R14 & 217	$\Rightarrow$	$\Delta\,\mathrm{m}(g_1, g_2) = (\mathrm{m}(g_2) - \mathrm{m}(g_1))\ \mathrm{mod}\ \mu_{\mathrm{m}}$
R16	R8 & R15	$\Rightarrow$	$\Delta\,\mathrm{m}(g_1, g_2) = ((\mathrm{m}(g_1) + \Delta\,\mathrm{m}(\Delta g))\ \mathrm{mod}\ \mu_{\mathrm{m}} - \mathrm{m}(g_1))\ \mathrm{mod}\ \mu_{\mathrm{m}}$

R17	R16 & 38	$\Rightarrow$	$\Delta\,\mathrm{m}\,(g_1, g_2) = (\mathrm{m}\,(g_1) + \Delta\,\mathrm{m}\,(\Delta g) - \mathrm{m}\,(g_1)) \bmod \mu_{\mathrm{m}}$
			$= \Delta\,\mathrm{m}\,(\Delta g) \bmod \mu_{\mathrm{m}}$
R18	R17, 44 & 218	$\Rightarrow$	$\Delta\,\mathrm{m}\,(g_1, g_2) = \Delta\,\mathrm{m}\,(\Delta g)$
R19	R3, R13 & R18	$\Rightarrow$	$\Delta\,\mathrm{g}\,(g_1, g_2) = [\Delta\,\mathrm{g_c}\,(\Delta g)\,, \Delta\,\mathrm{m}\,(\Delta g)]$
R20	R19 & 318	$\Rightarrow$	$\Delta\,\mathrm{g}\,(g_1, g_2) = \Delta g$
R21	R1 to R20	$\Rightarrow$	$\tau_{\mathrm{g}}\,(g_1, \Delta g) = g_2 \Rightarrow \Delta\,\mathrm{g}\,(g_1, g_2) = \Delta g$

Theorem 424 *If ψ is a pitch system and g_1 and g_2 are genera in ψ and Δg is a genus interval in ψ then*

$$\tau_{\mathrm{g}}\,(g_1, \Delta g) = g_2 \iff \Delta\,\mathrm{g}\,(g_1, g_2) = \Delta g$$

Proof

R1	423	$\Rightarrow$	$\tau_{\mathrm{g}}\,(g_1, \Delta g) = g_2 \Rightarrow \Delta\,\mathrm{g}\,(g_1, g_2) = \Delta g$
R2	421	$\Rightarrow$	$\Delta\,\mathrm{g}\,(g_1, g_2) = \Delta g \Rightarrow \tau_{\mathrm{g}}\,(g_1, \Delta g) = g_2$
R3	R1 & R2	$\Rightarrow$	$\tau_{\mathrm{g}}\,(g_1, \Delta g) = g_2 \iff \Delta\,\mathrm{g}\,(g_1, g_2) = \Delta g$

Theorem 425 *If ψ is a pitch system and Δg_1 and Δg_2 are genus intervals in ψ and g is a genus in ψ then*

$$(\tau_{\mathrm{g}}\,(g, \Delta g_1) = \tau_{\mathrm{g}}\,(g, \Delta g_2)) \Rightarrow (\Delta g_1 = \Delta g_2)$$

Proof

R1	Let		$\tau_{\mathrm{g}}\,(g, \Delta g_1) = g_2$
R2	Let		$\tau_{\mathrm{g}}\,(g, \Delta g_2) = g_2$
R3	R1 & 423	$\Rightarrow$	$\Delta\,\mathrm{g}\,(g, g_2) = \Delta g_1$
R4	R2 & 423	$\Rightarrow$	$\Delta\,\mathrm{g}\,(g, g_2) = \Delta g_2$
R5	R3 & R4	$\Rightarrow$	$\Delta g_1 = \Delta g_2$
R6	R1 to R5	$\Rightarrow$	$(\tau_{\mathrm{g}}\,(g, \Delta g_1) = \tau_{\mathrm{g}}\,(g, \Delta g_2)) \Rightarrow (\Delta g_1 = \Delta g_2)$

4.5.5 Transposing a chromatic pitch

Definition 426 (Definition of $\tau_{\mathrm{p_c}}\,(p_{\mathrm{c}}, \Delta p_{\mathrm{c}})$) *If ψ is a pitch system and $p_{\mathrm{c},1}$ and $p_{\mathrm{c},2}$ are chromatic pitches in ψ and Δp_{c} is a chromatic pitch interval in ψ then*

$$\Delta p_{\mathrm{c}} = \Delta\,\mathrm{p_c}\,(p_{\mathrm{c},1}, p_{\mathrm{c},2}) \Rightarrow \tau_{\mathrm{p_c}}\,(p_{\mathrm{c},1}, \Delta p_{\mathrm{c}}) = p_{\mathrm{c},2}$$

Theorem 427 (Formula for $\tau_{\mathrm{p_c}}(p_{\mathrm{c}}, \Delta p_{\mathrm{c}})$) *If ψ is a pitch system and p_{c} is a chromatic pitch in ψ and Δp_{c} is a chromatic pitch interval in ψ then*

$$\tau_{\mathrm{p_c}}(p_{\mathrm{c}}, \Delta p_{\mathrm{c}}) = p_{\mathrm{c}} + \Delta p_{\mathrm{c}}$$

Proof

R1	Let		$\Delta\,\mathrm{p_c}(p_{\mathrm{c}}, p_{\mathrm{c},2}) = \Delta p_{\mathrm{c}}$
R2	R1 & 426	$\Rightarrow$	$\tau_{\mathrm{p_c}}(p_{\mathrm{c}}, \Delta p_{\mathrm{c}}) = p_{\mathrm{c},2}$
R3	R1 & 236	$\Rightarrow$	$\Delta p_{\mathrm{c}} = p_{\mathrm{c},2} - p_{\mathrm{c}}$
			$\Rightarrow p_{\mathrm{c},2} = p_{\mathrm{c}} + \Delta p_{\mathrm{c}}$
R4	R2 & R3	$\Rightarrow$	$\tau_{\mathrm{p_c}}(p_{\mathrm{c}}, \Delta p_{\mathrm{c}}) = p_{\mathrm{c}} + \Delta p_{\mathrm{c}}$

Theorem 428 *If ψ is a pitch system and $p_{\mathrm{c},1}$ and $p_{\mathrm{c},2}$ are chromatic pitches in ψ and Δp_{c} is a chromatic pitch interval in ψ then*

$$\tau_{\mathrm{p_c}}(p_{\mathrm{c},1}, \Delta p_{\mathrm{c}}) = p_{\mathrm{c},2} \Rightarrow \Delta p_{\mathrm{c}} = \Delta\,\mathrm{p_c}(p_{\mathrm{c},1}, p_{\mathrm{c},2})$$

Proof

R1	Let		$\tau_{\mathrm{p_c}}(p_{\mathrm{c},1}, \Delta p_{\mathrm{c}}) = p_{\mathrm{c},2}$
R2	R1 & 427	$\Rightarrow$	$p_{\mathrm{c},2} = p_{\mathrm{c},1} + \Delta p_{\mathrm{c}}$
			$\Rightarrow \Delta p_{\mathrm{c}} = p_{\mathrm{c},2} - p_{\mathrm{c},1}$
R3	236	$\Rightarrow$	$\Delta\,\mathrm{p_c}(p_{\mathrm{c},1}, p_{\mathrm{c},2}) = p_{\mathrm{c},2} - p_{\mathrm{c},1}$
R4	R2 & R3	$\Rightarrow$	$\Delta p_{\mathrm{c}} = \Delta\,\mathrm{p_c}(p_{\mathrm{c},1}, p_{\mathrm{c},2})$
R5	R1 to R4	$\Rightarrow$	$\tau_{\mathrm{p_c}}(p_{\mathrm{c},1}, \Delta p_{\mathrm{c}}) = p_{\mathrm{c},2} \Rightarrow \Delta p_{\mathrm{c}} = \Delta\,\mathrm{p_c}(p_{\mathrm{c},1}, p_{\mathrm{c},2})$

Theorem 429 *If ψ is a pitch system and $p_{\mathrm{c},1}$ and $p_{\mathrm{c},2}$ are chromatic pitches in ψ and Δp_{c} is a chromatic pitch interval in ψ then*

$$\tau_{\mathrm{p_c}}(p_{\mathrm{c},1}, \Delta p_{\mathrm{c}}) = p_{\mathrm{c},2} \iff \Delta p_{\mathrm{c}} = \Delta\,\mathrm{p_c}(p_{\mathrm{c},1}, p_{\mathrm{c},2})$$

Proof

R1	426	$\Rightarrow$	$\Delta p_{\mathrm{c}} = \Delta\,\mathrm{p_c}(p_{\mathrm{c},1}, p_{\mathrm{c},2}) \Rightarrow \tau_{\mathrm{p_c}}(p_{\mathrm{c},1}, \Delta p_{\mathrm{c}}) = p_{\mathrm{c},2}$
R2	428	$\Rightarrow$	$\tau_{\mathrm{p_c}}(p_{\mathrm{c},1}, \Delta p_{\mathrm{c}}) = p_{\mathrm{c},2} \Rightarrow \Delta p_{\mathrm{c}} = \Delta\,\mathrm{p_c}(p_{\mathrm{c},1}, p_{\mathrm{c},2})$
R3	R1 & R2	$\Rightarrow$	$\tau_{\mathrm{p_c}}(p_{\mathrm{c},1}, \Delta p_{\mathrm{c}}) = p_{\mathrm{c},2} \iff \Delta p_{\mathrm{c}} = \Delta\,\mathrm{p_c}(p_{\mathrm{c},1}, p_{\mathrm{c},2})$

Theorem 430 *If ψ is a pitch system and $\Delta p_{\mathrm{c},1}$ and $\Delta p_{\mathrm{c},2}$ are chromatic pitch intervals in ψ and p_{c} is a chromatic pitch in ψ then*

$$(\tau_{\mathrm{p_c}}(p_{\mathrm{c}}, \Delta p_{\mathrm{c},1}) = \tau_{\mathrm{p_c}}(p_{\mathrm{c}}, \Delta p_{\mathrm{c},2})) \Rightarrow (\Delta p_{\mathrm{c},1} = \Delta p_{\mathrm{c},2})$$

Proof

R1	427	$\Rightarrow$	$\tau_{\mathrm{p_c}}(p_{\mathrm{c}}, \Delta p_{\mathrm{c},1}) = p_{\mathrm{c}} + \Delta p_{\mathrm{c},1}$
R2	427	$\Rightarrow$	$\tau_{\mathrm{p_c}}(p_{\mathrm{c}}, \Delta p_{\mathrm{c},2}) = p_{\mathrm{c}} + \Delta p_{\mathrm{c},2}$
R3	R1 & R2	$\Rightarrow$	$(\tau_{\mathrm{p_c}}(p_{\mathrm{c}}, \Delta p_{\mathrm{c},1}) = \tau_{\mathrm{p_c}}(p_{\mathrm{c}}, \Delta p_{\mathrm{c},2})) \Rightarrow (p_{\mathrm{c}} + \Delta p_{\mathrm{c},2} = p_{\mathrm{c}} + \Delta p_{\mathrm{c},1})$
			$\Rightarrow (\Delta p_{\mathrm{c},2} = \Delta p_{\mathrm{c},1})$

4.5.6 Transposing a morphetic pitch

Definition 431 (Definition of $\tau_{\mathrm{p_m}}(p_{\mathrm{m}}, \Delta p_{\mathrm{m}})$) *If ψ is a pitch system and $p_{\mathrm{m},1}$ and $p_{\mathrm{m},2}$ are morphetic pitches in ψ and Δp_{m} is a morphetic pitch interval in ψ then*

$$\Delta p_{\mathrm{m}} = \Delta\,\mathrm{p_m}(p_{\mathrm{m},1}, p_{\mathrm{m},2}) \Rightarrow \tau_{\mathrm{p_m}}(p_{\mathrm{m},1}, \Delta p_{\mathrm{m}}) = p_{\mathrm{m},2}$$

Theorem 432 (Formula for $\tau_{\mathrm{p_m}}(p_{\mathrm{m}}, \Delta p_{\mathrm{m}})$) *If ψ is a pitch system and p_{m} is a morphetic pitch in ψ and Δp_{m} is a morphetic pitch interval in ψ then*

$$\tau_{\mathrm{p_m}}(p_{\mathrm{m}}, \Delta p_{\mathrm{m}}) = p_{\mathrm{m}} + \Delta p_{\mathrm{m}}$$

Proof

R1	Let		$\Delta\,\mathrm{p_m}(p_{\mathrm{m}}, p_{\mathrm{m},2}) = \Delta p_{\mathrm{m}}$
R2	R1 & 431	$\Rightarrow$	$\tau_{\mathrm{p_m}}(p_{\mathrm{m}}, \Delta p_{\mathrm{m}}) = p_{\mathrm{m},2}$
R3	R1 & 240	$\Rightarrow$	$\Delta p_{\mathrm{m}} = p_{\mathrm{m},2} - p_{\mathrm{m}}$
			$\Rightarrow p_{\mathrm{m},2} = p_{\mathrm{m}} + \Delta p_{\mathrm{m}}$
R4	R2 & R3	$\Rightarrow$	$\tau_{\mathrm{p_m}}(p_{\mathrm{m}}, \Delta p_{\mathrm{m}}) = p_{\mathrm{m}} + \Delta p_{\mathrm{m}}$

Theorem 433 *If ψ is a pitch system and $p_{\mathrm{m},1}$ and $p_{\mathrm{m},2}$ are morphetic pitches in ψ and Δp_{m} is a morphetic pitch interval in ψ then*

$$\tau_{\mathrm{p_m}}(p_{\mathrm{m},1}, \Delta p_{\mathrm{m}}) = p_{\mathrm{m},2} \Rightarrow \Delta p_{\mathrm{m}} = \Delta\,\mathrm{p_m}(p_{\mathrm{m},1}, p_{\mathrm{m},2})$$

Proof

R1	Let		$\tau_{\mathrm{p_m}}(p_{\mathrm{m},1}, \Delta p_{\mathrm{m}}) = p_{\mathrm{m},2}$
R2	R1 & 432	$\Rightarrow$	$p_{\mathrm{m},2} = p_{\mathrm{m},1} + \Delta p_{\mathrm{m}}$
			$\Rightarrow \Delta p_{\mathrm{m}} = p_{\mathrm{m},2} - p_{\mathrm{m},1}$
R3	240	$\Rightarrow$	$\Delta\,\mathrm{p_m}(p_{\mathrm{m},1}, p_{\mathrm{m},2}) = p_{\mathrm{m},2} - p_{\mathrm{m},1}$
R4	R2 & R3	$\Rightarrow$	$\Delta p_{\mathrm{m}} = \Delta\,\mathrm{p_m}(p_{\mathrm{m},1}, p_{\mathrm{m},2})$
R5	R1 to R4	$\Rightarrow$	$\tau_{\mathrm{p_m}}(p_{\mathrm{m},1}, \Delta p_{\mathrm{m}}) = p_{\mathrm{m},2} \Rightarrow \Delta p_{\mathrm{m}} = \Delta\,\mathrm{p_m}(p_{\mathrm{m},1}, p_{\mathrm{m},2})$

Theorem 434 *If ψ is a pitch system and $p_{\mathrm{m},1}$ and $p_{\mathrm{m},2}$ are morphetic pitches in ψ and Δp_{m} is a morphetic pitch interval in ψ then*

$$\tau_{\mathrm{p_m}}(p_{\mathrm{m},1}, \Delta p_{\mathrm{m}}) = p_{\mathrm{m},2} \iff \Delta p_{\mathrm{m}} = \Delta\,\mathrm{p_m}(p_{\mathrm{m},1}, p_{\mathrm{m},2})$$

Proof

R1	431	$\Rightarrow$	$\Delta p_{\mathrm{m}} = \Delta\,\mathrm{p_m}(p_{\mathrm{m},1}, p_{\mathrm{m},2}) \Rightarrow \tau_{\mathrm{p_m}}(p_{\mathrm{m},1}, \Delta p_{\mathrm{m}}) = p_{\mathrm{m},2}$
R2	433	$\Rightarrow$	$\tau_{\mathrm{p_m}}(p_{\mathrm{m},1}, \Delta p_{\mathrm{m}}) = p_{\mathrm{m},2} \Rightarrow \Delta p_{\mathrm{m}} = \Delta\,\mathrm{p_m}(p_{\mathrm{m},1}, p_{\mathrm{m},2})$
R3	R1 & R2	$\Rightarrow$	$\tau_{\mathrm{p_m}}(p_{\mathrm{m},1}, \Delta p_{\mathrm{m}}) = p_{\mathrm{m},2} \iff \Delta p_{\mathrm{m}} = \Delta\,\mathrm{p_m}(p_{\mathrm{m},1}, p_{\mathrm{m},2})$

Theorem 435 *If ψ is a pitch system and $\Delta p_{\mathrm{m},1}$ and $\Delta p_{\mathrm{m},2}$ are morphetic pitch intervals in ψ and p_{m} is a morphetic pitch in ψ then*

$$(\tau_{\mathrm{p_m}}(p_{\mathrm{m}}, \Delta p_{\mathrm{m},1}) = \tau_{\mathrm{p_m}}(p_{\mathrm{m}}, \Delta p_{\mathrm{m},2})) \Rightarrow (\Delta p_{\mathrm{m},1} = \Delta p_{\mathrm{m},2})$$

Proof

R1	432	$\Rightarrow$	$\tau_{\mathrm{p_m}}(p_{\mathrm{m}}, \Delta p_{\mathrm{m},1}) = p_{\mathrm{m}} + \Delta p_{\mathrm{m},1}$
R2	432	$\Rightarrow$	$\tau_{\mathrm{p_m}}(p_{\mathrm{m}}, \Delta p_{\mathrm{m},2}) = p_{\mathrm{m}} + \Delta p_{\mathrm{m},2}$
R3	R1 & R2	$\Rightarrow$	$(\tau_{\mathrm{p_m}}(p_{\mathrm{m}}, \Delta p_{\mathrm{m},1}) = \tau_{\mathrm{p_m}}(p_{\mathrm{m}}, \Delta p_{\mathrm{m},2})) \Rightarrow (p_{\mathrm{m}} + \Delta p_{\mathrm{m},2} = p_{\mathrm{m}} + \Delta p_{\mathrm{m},1})$
			$\Rightarrow (\Delta p_{\mathrm{m},2} = \Delta p_{\mathrm{m},1})$

4.5.7 Transposing a frequency

Definition 436 (Definition of $\tau_{\mathrm{f}}(f, \Delta f)$) *If ψ is a pitch system and f_1 and f_2 are frequencies in ψ and Δf is a frequency interval in ψ then*

$$\Delta f = \Delta\,\mathrm{f}(f_1, f_2) \Rightarrow \tau_{\mathrm{f}}(f_1, \Delta f) = f_2$$

Theorem 437 (Formula for $\tau_{\mathrm{f}}(f, \Delta f)$) *If ψ is a pitch system and f is a frequency in ψ and Δf is a frequency interval in ψ then*

$$\tau_{\mathrm{f}}(f, \Delta f) = f \times \Delta f$$

Proof

R1	Let		$\Delta\mathrm{f}(f, f_2) = \Delta f$
R2	R1 & 436	$\Rightarrow$	$\tau_{\mathrm{f}}(f, \Delta f) = f_2$
R3	R1 & 242	$\Rightarrow$	$\Delta f = \frac{f_2}{f}$
			$\Rightarrow f_2 = f \times \Delta f$
R4	R2 & R3	$\Rightarrow$	$\tau_{\mathrm{f}}(f, \Delta f) = f \times \Delta f$

Theorem 438 *If ψ is a pitch system and f_1 and f_2 are frequencies in ψ and Δf is a frequency interval in ψ then*

$$\tau_{\mathrm{f}}(f_1, \Delta f) = f_2 \Rightarrow \Delta f = \Delta\mathrm{f}(f_1, f_2)$$

Proof

R1	Let		$\tau_{\mathrm{f}}(f_1, \Delta f) = f_2$
R2	R1 & 437	$\Rightarrow$	$f_2 = f_1 \times \Delta f$
			$\Rightarrow \Delta f = \frac{f_2}{f_1}$
R3	242	$\Rightarrow$	$\Delta\mathrm{f}(f_1, f_2) = \frac{f_2}{f_1}$
R4	R2 & R3	$\Rightarrow$	$\Delta f = \Delta\mathrm{f}(f_1, f_2)$
R5	R1 to R4	$\Rightarrow$	$\tau_{\mathrm{f}}(f_1, \Delta f) = f_2 \Rightarrow \Delta f = \Delta\mathrm{f}(f_1, f_2)$

Theorem 439 *If ψ is a pitch system and f_1 and f_2 are frequencies in ψ and Δf is a frequency interval in ψ then*

$$\tau_{\mathrm{f}}(f_1, \Delta f) = f_2 \iff \Delta f = \Delta\mathrm{f}(f_1, f_2)$$

Proof

R1	436	$\Rightarrow$	$\Delta f = \Delta\mathrm{f}(f_1, f_2) \Rightarrow \tau_{\mathrm{f}}(f_1, \Delta f) = f_2$
R2	438	$\Rightarrow$	$\tau_{\mathrm{f}}(f_1, \Delta f) = f_2 \Rightarrow \Delta f = \Delta\mathrm{f}(f_1, f_2)$
R3	R1 & R2	$\Rightarrow$	$\tau_{\mathrm{f}}(f_1, \Delta f) = f_2 \iff \Delta f = \Delta\mathrm{f}(f_1, f_2)$

Theorem 440 *If ψ is a pitch system and Δf_1 and Δf_2 are frequency intervals in ψ and f is a frequency in ψ then*

$$(\tau_{\mathrm{f}}(f, \Delta f_1) = \tau_{\mathrm{f}}(f, \Delta f_2)) \Rightarrow (\Delta f_1 = \Delta f_2)$$

Proof

R1	437	$\Rightarrow$	$\tau_{\mathrm{f}}(f, \Delta f_1) = f \times \Delta f_1$
R2	437	$\Rightarrow$	$\tau_{\mathrm{f}}(f, \Delta f_2) = f \times \Delta f_2$
R3	R1 & R2	$\Rightarrow$	$(\tau_{\mathrm{f}}(f, \Delta f_1) = \tau_{\mathrm{f}}(f, \Delta f_2)) \Rightarrow (f \times \Delta f_2 = f \times \Delta f_1)$
			$\Rightarrow (\Delta f_2 = \Delta f_1)$

4.5.8 Transposing a pitch

Definition 441 (Definition of $\tau_{\mathrm{p}}(p, \Delta p)$) *If ψ is a pitch system and p_1 and p_2 are pitches in ψ and Δp is a pitch interval in ψ then*

$$\Delta p = \Delta\,\mathrm{p}(p_1, p_2) \Rightarrow \tau_{\mathrm{p}}(p_1, \Delta p) = p_2$$

Theorem 442 (Formula for $\tau_{\mathrm{p}}(p, \Delta p)$) *If ψ is a pitch system and p is a pitch in ψ and Δp is a pitch interval in ψ then*

$$\tau_{\mathrm{p}}(p, \Delta p) = [\tau_{\mathrm{p_c}}(\mathrm{p_c}(p), \Delta\,\mathrm{p_c}(\Delta p)), \tau_{\mathrm{p_m}}(\mathrm{p_m}(p), \Delta\,\mathrm{p_m}(\Delta p))]$$

Proof

R1	Let		$\Delta\,\mathrm{p}\,(p, p_2) = \Delta p$
R2	R1 & 441	$\Rightarrow$	$\tau_{\mathrm{p}}\,(p, \Delta p) = p_2$
R3	R1 & 265	$\Rightarrow$	$\Delta p = [\Delta\,\mathrm{p_c}\,(p, p_2)\,, \Delta\,\mathrm{p_m}\,(p, p_2)]$
R4	R3 & 267	$\Rightarrow$	$\Delta\,\mathrm{p_c}\,(\Delta p) = \Delta\,\mathrm{p_c}\,(p, p_2)$
R5	R3 & 269	$\Rightarrow$	$\Delta\,\mathrm{p_m}\,(\Delta p) = \Delta\,\mathrm{p_m}\,(p, p_2)$
R6	R4 & 260	$\Rightarrow$	$\Delta\,\mathrm{p_c}\,(\Delta p) = \mathrm{p_c}\,(p_2) - \mathrm{p_c}\,(p)$
			$\Rightarrow \mathrm{p_c}\,(p_2) = \mathrm{p_c}\,(p) + \Delta\,\mathrm{p_c}\,(\Delta p)$
R7	R6 & 427	$\Rightarrow$	$\mathrm{p_c}\,(p_2) = \tau_{\mathrm{p_c}}\,(\mathrm{p_c}\,(p)\,, \Delta\,\mathrm{p_c}\,(\Delta p))$
R8	R5 & 262	$\Rightarrow$	$\Delta\,\mathrm{p_m}\,(\Delta p) = \mathrm{p_m}\,(p_2) - \mathrm{p_m}\,(p)$
			$\Rightarrow \mathrm{p_m}\,(p_2) = \mathrm{p_m}\,(p) + \Delta\,\mathrm{p_m}\,(\Delta p)$
R9	R8 & 432	$\Rightarrow$	$\mathrm{p_m}\,(p_2) = \tau_{\mathrm{p_m}}\,(\mathrm{p_m}\,(p)\,, \Delta\,\mathrm{p_m}\,(\Delta p))$
R10	R7, R9 & 65	$\Rightarrow$	$p_2 = [\tau_{\mathrm{p_c}}\,(\mathrm{p_c}\,(p)\,, \Delta\,\mathrm{p_c}\,(\Delta p))\,, \tau_{\mathrm{p_m}}\,(\mathrm{p_m}\,(p)\,, \Delta\,\mathrm{p_m}\,(\Delta p))]$
R11	R2 & R10	$\Rightarrow$	$\tau_{\mathrm{p}}\,(p, \Delta p) = [\tau_{\mathrm{p_c}}\,(\mathrm{p_c}\,(p)\,, \Delta\,\mathrm{p_c}\,(\Delta p))\,, \tau_{\mathrm{p_m}}\,(\mathrm{p_m}\,(p)\,, \Delta\,\mathrm{p_m}\,(\Delta p))]$

Theorem 443 *If ψ is a pitch system and p_1 and p_2 are pitches in ψ and Δp is a pitch interval in ψ then*

$$\tau_{\mathrm{p}}\,(p_1, \Delta p) = p_2 \Rightarrow \Delta p = \Delta\,\mathrm{p}\,(p_1, p_2)$$

Proof

R1	Let		$\tau_{\mathrm{p}}(p_1, \Delta p) = p_2$
R2	R1 & 442	$\Rightarrow$	$p_2 = [\tau_{\mathrm{p_c}}(\mathrm{p_c}(p_1), \Delta\,\mathrm{p_c}(\Delta p)), \tau_{\mathrm{p_m}}(\mathrm{p_m}(p_1), \Delta\,\mathrm{p_m}(\Delta p))]$
R3	265	$\Rightarrow$	$\Delta\,\mathrm{p}(p_1, p_2) = [\Delta\,\mathrm{p_c}(p_1, p_2), \Delta\,\mathrm{p_m}(p_1, p_2)]$
R4	270	$\Rightarrow$	$\Delta p = [\Delta\,\mathrm{p_c}(\Delta p_{\mathrm{c}}), \Delta\,\mathrm{p_m}(\Delta p)]$
R5	427	$\Rightarrow$	$\tau_{\mathrm{p_c}}(\mathrm{p_c}(p_1), \Delta\,\mathrm{p}(\Delta p)) = \mathrm{p_c}(p_1) + \Delta\,\mathrm{p_c}(\Delta p)$
R6	432	$\Rightarrow$	$\tau_{\mathrm{p_m}}(\mathrm{p_m}(p_1), \Delta\,\mathrm{p}(\Delta p)) = \mathrm{p_m}(p_1) + \Delta\,\mathrm{p_m}(\Delta p)$
R7	R5 & 65	$\Rightarrow$	$p_2 = [\mathrm{p_c}(p_2), \mathrm{p_m}(p_2)]$
R8	R2, R5 & R7	$\Rightarrow$	$\mathrm{p_c}(p_2) = \mathrm{p_c}(p_1) + \Delta\,\mathrm{p_c}(\Delta p)$
			$\Rightarrow \Delta\,\mathrm{p_c}(\Delta p) = \mathrm{p_c}(p_2) - \mathrm{p_c}(p_1)$
R9	R8 & 236	$\Rightarrow$	$\Delta\,\mathrm{p_c}(\mathrm{p_c}(p_1), \mathrm{p_c}(p_2)) = \Delta\,\mathrm{p_c}(\Delta p)$
R10	R2, R6 & R7	$\Rightarrow$	$\mathrm{p_m}(p_2) = \mathrm{p_m}(p_1) + \Delta\,\mathrm{p_m}(\Delta p)$
			$\Rightarrow \Delta\,\mathrm{p_m}(\Delta p) = \mathrm{p_m}(p_2) - \mathrm{p_m}(p_1)$
R11	R10 & 240	$\Rightarrow$	$\Delta\,\mathrm{p_m}(\mathrm{p_m}(p_1), \mathrm{p_m}(p_2)) = \Delta\,\mathrm{p_m}(\Delta p)$
R12	R4, R9 & R11	$\Rightarrow$	$\Delta p = [\Delta\,\mathrm{p_c}(\mathrm{p_c}(p_1), \mathrm{p_c}(p_2)), \Delta\,\mathrm{p_m}(\mathrm{p_m}(p_1), \mathrm{p_m}(p_2))]$
R13	R12, 259 & 261	$\Rightarrow$	$\Delta p = [\Delta\,\mathrm{p_c}(p_1, p_2), \Delta\,\mathrm{p_m}(p_1, p_2)]$
R14	R3 & R13	$\Rightarrow$	$\Delta p = \Delta\,\mathrm{p}(p_1, p_2)$
R15	R1 to R14	$\Rightarrow$	$\tau_{\mathrm{p}}(p_1, \Delta p) = p_2 \Rightarrow \Delta p = \Delta\,\mathrm{p}(p_1, p_2)$

Theorem 444 *If ψ is a pitch system and p_1 and p_2 are pitches in ψ and Δp is a pitch interval in ψ then*

$$\tau_{\mathrm{p}}(p_1, \Delta p) = p_2 \iff \Delta p = \Delta\,\mathrm{p}(p_1, p_2)$$

Proof

R1	441	$\Rightarrow$	$\Delta p = \Delta\,\mathrm{p}\,(p_1, p_2) \Rightarrow \tau_{\mathrm{p}}\,(p_1, \Delta p) = p_2$
R2	443	$\Rightarrow$	$\tau_{\mathrm{p}}\,(p_1, \Delta p) = p_2 \Rightarrow \Delta p = \Delta\,\mathrm{p}\,(p_1, p_2)$
R3	R1 & R2	$\Rightarrow$	$\tau_{\mathrm{p}}\,(p_1, \Delta p) = p_2 \iff \Delta p = \Delta\,\mathrm{p}\,(p_1, p_2)$

Theorem 445 *If ψ is a pitch system and Δp_1 and Δp_2 are pitch intervals in ψ and p is a pitch in ψ then*

$$(\tau_{\mathrm{p}}\,(p, \Delta p_1) = \tau_{\mathrm{p}}\,(p, \Delta p_2)) \Rightarrow (\Delta p_1 = \Delta p_2)$$

Proof

R1	Let		$\tau_{\mathrm{p}}\,(p, \Delta p_1) = \tau_{\mathrm{p}}\,(p, \Delta p_2)$
R2	R1 & 443	$\Rightarrow$	$\Delta p_1 = \Delta\,\mathrm{p}\,(p, \tau_{\mathrm{p}}\,(p, \Delta p_2))$
R3	R2 & 442	$\Rightarrow$	$\Delta p_1 = \Delta\,\mathrm{p}\,(p, [\tau_{\mathrm{p_c}}\,(\mathrm{p_c}\,(p)\,, \Delta\,\mathrm{p_c}\,(\Delta p_2))\,, \tau_{\mathrm{p_m}}\,(\mathrm{p_m}\,(p)\,, \Delta\,\mathrm{p_m}\,(\Delta p_2))])$
R4	R3, 427 & 432	$\Rightarrow$	$\Delta p_1 = \Delta\,\mathrm{p}\,(p, [\mathrm{p_c}\,(p) + \Delta\,\mathrm{p_c}\,(\Delta p_2)\,, \mathrm{p_m}\,(p) + \Delta\,\mathrm{p_m}\,(\Delta p_2)])$
R5	R4 & 265	$\Rightarrow$	$\Delta p_1 = \begin{bmatrix} \Delta\,\mathrm{p_c}\,(p, [\mathrm{p_c}\,(p) + \Delta\,\mathrm{p_c}\,(\Delta p_2)\,, \mathrm{p_m}\,(p) + \Delta\,\mathrm{p_m}\,(\Delta p_2)])\,, \\ \Delta\,\mathrm{p_m}\,(p, [\mathrm{p_c}\,(p) + \Delta\,\mathrm{p_c}\,(\Delta p_2)\,, \mathrm{p_m}\,(p) + \Delta\,\mathrm{p_m}\,(\Delta p_2)]) \end{bmatrix}$
R6	R5, 260, 262, 63 & 64	$\Rightarrow$	$\Delta p_1 = \begin{bmatrix} \mathrm{p_c}\,(p) + \Delta\,\mathrm{p_c}\,(\Delta p_2) - \mathrm{p_c}\,(p)\,, \\ \mathrm{p_m}\,(p) + \Delta\,\mathrm{p_m}\,(\Delta p_2) - \mathrm{p_m}\,(p) \end{bmatrix}$
			$\Rightarrow \Delta p_1 = [\Delta\,\mathrm{p_c}\,(\Delta p_2)\,, \Delta\,\mathrm{p_m}\,(\Delta p_2)]$
R7	270	$\Rightarrow$	$\Delta p_2 = [\Delta\,\mathrm{p_c}\,(\Delta p_2)\,, \Delta\,\mathrm{p_m}\,(\Delta p_2)]$
R8	R6 & R7	$\Rightarrow$	$\Delta p_1 = \Delta p_2$
R9	R1 to R8	$\Rightarrow$	$(\tau_{\mathrm{p}}\,(p, \Delta p_1) = \tau_{\mathrm{p}}\,(p, \Delta p_2)) \Rightarrow (\Delta p_1 = \Delta p_2)$

Theorem 446 *If ψ is a pitch system and p is a pitch in ψ and Δp is a pitch interval in ψ then*

$$\tau_{\mathrm{p}}\,(p, \Delta p) = [\mathrm{p_c}\,(p) + \Delta\,\mathrm{p_c}\,(\Delta p)\,, \mathrm{p_m}\,(p) + \Delta\,\mathrm{p_m}\,(\Delta p)]$$

Proof

R1	442	$\Rightarrow$	$\tau_{\mathrm{p}}\,(p, \Delta p) = [\tau_{\mathrm{p_c}}\,(\mathrm{p_c}\,(p)\,, \Delta\,\mathrm{p_c}\,(\Delta p))\,, \tau_{\mathrm{p_m}}\,(\mathrm{p_m}\,(p)\,, \Delta\,\mathrm{p_m}\,(\Delta p))]$
R2	R1, 427 & 432	$\Rightarrow$	$\tau_{\mathrm{p}}\,(p, \Delta p) = [\mathrm{p_c}\,(p) + \Delta\,\mathrm{p_c}\,(\Delta p)\,, \mathrm{p_m}\,(p) + \Delta\,\mathrm{p_m}\,(\Delta p)]$

4.6 Summation, inversion and exponentiation of *MIPS* intervals

4.6.1 Summation, inversion and exponentiation of chroma intervals

Summation of chroma intervals

Definition 447 (Definition of $\sigma_{\mathrm{c}}(\Delta c_1, \Delta c_2, \ldots, \Delta c_n)$) *If*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

is a pitch system and

$$\Delta c_1, \Delta c_2, \ldots, \Delta c_n$$

is a collection of chroma intervals in ψ then

$$\sigma_{\mathrm{c}}(\Delta c_1, \Delta c_2, \ldots, \Delta c_n) = \left(\sum_{k=1}^{n} \Delta c_k\right) \bmod \mu_{\mathrm{c}}$$

Theorem 448 *If ψ is a pitch system and*

$$\Delta c_1, \Delta c_2, \ldots, \Delta c_n$$

is a collection of chroma intervals in ψ and c is a chroma in ψ then

$$\tau_{\mathrm{c}}(c, \sigma_{\mathrm{c}}(\Delta c_1, \Delta c_2, \ldots, \Delta c_n)) = \tau_{\mathrm{c}}(\ldots \tau_{\mathrm{c}}(\tau_{\mathrm{c}}(c, \Delta c_1), \Delta c_2) \ldots, \Delta c_n)$$

Proof

R1 407 $\Rightarrow$ $\tau_{\mathrm{c}}(\ldots \tau_{\mathrm{c}}(\tau_{\mathrm{c}}(c, \Delta c_1), \Delta c_2) \ldots, \Delta c_n)$

$= \tau_{\mathrm{c}}(\ldots \tau_{\mathrm{c}}((c + \Delta c_1) \bmod \mu_{\mathrm{c}}, \Delta c_2) \ldots, \Delta c_n)$

$= (\ldots((c + \Delta c_1) \bmod \mu_{\mathrm{c}} + \Delta c_2) \bmod \mu_{\mathrm{c}} \ldots + \Delta c_n) \bmod \mu_{\mathrm{c}}$

R2 R1 & 38 $\Rightarrow$ $\tau_{\mathrm{c}}(\ldots \tau_{\mathrm{c}}(\tau_{\mathrm{c}}(c, \Delta c_1), \Delta c_2) \ldots, \Delta c_n)$

$= (c + \Delta c_1 + \Delta c_2 + \ldots + \Delta c_n) \bmod \mu_{\mathrm{c}}$

$= (c + \sum_{k=1}^{n} \Delta c_k) \bmod \mu_{\mathrm{c}}$

R3 R2 & 38 $\Rightarrow$ $\tau_{\mathrm{c}}(\ldots \tau_{\mathrm{c}}(\tau_{\mathrm{c}}(c, \Delta c_1), \Delta c_2) \ldots, \Delta c_n)$

$= (c + (\sum_{k=1}^{n} \Delta c_k) \bmod \mu_{\mathrm{c}}) \bmod \mu_{\mathrm{c}}$

R4 R3 & 447 $\Rightarrow$ $\tau_{\mathrm{c}}(\ldots \tau_{\mathrm{c}}(\tau_{\mathrm{c}}(c, \Delta c_1), \Delta c_2) \ldots, \Delta c_n)$

$= (c + \sigma_{\mathrm{c}}(\Delta c_1, \Delta c_2, \ldots, \Delta c_n)) \bmod \mu_{\mathrm{c}}$

R5 R4 & 407 $\Rightarrow$ $\tau_{\mathrm{c}}(\ldots \tau_{\mathrm{c}}(\tau_{\mathrm{c}}(c, \Delta c_1), \Delta c_2) \ldots, \Delta c_n)$

$= \tau_{\mathrm{c}}(c, \sigma_{\mathrm{c}}(\Delta c_1, \Delta c_2, \ldots, \Delta c_n))$

Inversion of chroma intervals

Definition 449 (Definition of $\iota_{\mathrm{c}}(\Delta c)$) *If ψ is a pitch system and Δc is a chroma interval in ψ and c is a chroma in ψ then $\iota_{\mathrm{c}}(\Delta c)$ is the chroma interval that satisfies the following equation*

$$\tau_{\mathrm{c}}(\tau_{\mathrm{c}}(c, \Delta c), \iota_{\mathrm{c}}(\Delta c)) = c$$

Definition 450 (Inversional equivalence of chroma intervals) *If ψ is a pitch system and Δc_1 and Δc_2 are chroma intervals in ψ then Δc_1 and Δc_2 are* inversionally equivalent *if and only if*

$$(\iota_{\mathrm{c}}(\Delta c_1) = \Delta c_2) \vee (\Delta c_1 = \Delta c_2)$$

The fact that two chroma intervals are inversionally equivalent is denoted as follows:

$$\Delta c_1 \equiv_{\iota} \Delta c_2$$

Theorem 451 *If*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

is a pitch system and Δc is a chroma interval in ψ and c is a chroma in ψ then

$$\iota_{\mathrm{c}}(\Delta c) = (-\Delta c) \bmod \mu_{\mathrm{c}}$$

Proof

R1	449	$\Rightarrow$	$\tau_{\mathrm{c}}(\tau_{\mathrm{c}}(c, \Delta c), \iota_{\mathrm{c}}(\Delta c)) = c$
R2	407	$\Rightarrow$	$\tau_{\mathrm{c}}(\tau_{\mathrm{c}}(c, \Delta c), (-\Delta c) \bmod \mu_{\mathrm{c}})$
			$= \tau_{\mathrm{c}}((c + \Delta c) \bmod \mu_{\mathrm{c}}, (-\Delta c) \bmod \mu_{\mathrm{c}})$
			$= ((c + \Delta c) \bmod \mu_{\mathrm{c}} + (-\Delta c) \bmod \mu_{\mathrm{c}}) \bmod \mu_{\mathrm{c}}$
R3	R2 &34	$\Rightarrow$	$\tau_{\mathrm{c}}(\tau_{\mathrm{c}}(c, \Delta c), (-\Delta c) \bmod \mu_{\mathrm{c}})$
			$= (c + \Delta c - \Delta c) \bmod \mu_{\mathrm{c}}$
			$= c \bmod \mu_{\mathrm{c}}$
R4	72	$\Rightarrow$	$(0 \leq c < \mu_{\mathrm{c}}) \wedge (c \in \mathbb{Z})$
R5	R3, R4 & 44	$\Rightarrow$	$\tau_{\mathrm{c}}(\tau_{\mathrm{c}}(c, \Delta c), (-\Delta c) \bmod \mu_{\mathrm{c}}) = c$
R6	R5 & R1	$\Rightarrow$	$\tau_{\mathrm{c}}(\tau_{\mathrm{c}}(c, \Delta c), (-\Delta c) \bmod \mu_{\mathrm{c}}) = \tau_{\mathrm{c}}(\tau_{\mathrm{c}}(c, \Delta c), \iota_{\mathrm{c}}(\Delta c))$
R7	R6 & 410	$\Rightarrow$	$\iota_{\mathrm{c}}(\Delta c) = (-\Delta c) \bmod \mu_{\mathrm{c}}$

Theorem 452 *If ψ is a pitch system and Δc, Δc_1 and Δc_2 are chroma intervals in ψ then*

$$(\Delta c_1 = \iota_{\mathrm{c}}(\Delta c)) \wedge (\Delta c_2 = \iota_{\mathrm{c}}(\Delta c)) \Rightarrow (\Delta c_1 = \Delta c_2)$$

Proof

R1	Let		$\Delta c_1 = \iota_{\mathrm{c}}(\Delta c)$
R2	Let		$\Delta c_2 = \iota_{\mathrm{c}}(\Delta c)$
R3	R1 & 449	$\Rightarrow$	$\tau_{\mathrm{c}}(\tau_{\mathrm{c}}(c, \Delta c), \Delta c_1) = c$
R4	R2 & 449	$\Rightarrow$	$\tau_{\mathrm{c}}(\tau_{\mathrm{c}}(c, \Delta c), \Delta c_2) = c$
R5	R3 & R4	$\Rightarrow$	$\tau_{\mathrm{c}}(\tau_{\mathrm{c}}(c, \Delta c), \Delta c_1) = \tau_{\mathrm{c}}(\tau_{\mathrm{c}}(c, \Delta c), \Delta c_2)$
R6	R5 & 410	$\Rightarrow$	$\Delta c_1 = \Delta c_2$
R7	R1 to R6	$\Rightarrow$	$(\Delta c_1 = \iota_{\mathrm{c}}(\Delta c)) \wedge (\Delta c_2 = \iota_{\mathrm{c}}(\Delta c)) \Rightarrow (\Delta c_1 = \Delta c_2)$

Exponentiation of chroma intervals

Definition 453 (Definition of $\epsilon_{\mathrm{c},n}(\Delta c)$) *Given that:*

1. *ψ is a pitch system;*
2. *c is a chroma in ψ;*
3. *Δc is a chroma interval in ψ;*
4. *n is an integer;*
5. *k is an integer and $1 \leq k \leq \mathrm{abs}(n)$;*
6. *$\Delta c_{1,k} = \Delta c$ for all k; and*
7. *$\Delta c_{2,k} = \iota_{\mathrm{c}}(\Delta c)$ for all k;*

then $\epsilon_{\mathrm{c},n}(\Delta c)$ is any chroma interval that satisfies the following equation:

$$\tau_{\mathrm{c}}(c, \epsilon_{\mathrm{c},n}(\Delta c)) = \begin{cases} \tau_{\mathrm{c}}(c, \sigma_{\mathrm{c}}(\Delta c_{1,1}, \Delta c_{1,2}, \ldots \Delta c_{1,n})) & \text{if} \quad n > 0 \\ c & \text{if} \quad n = 0 \\ \tau_{\mathrm{c}}(c, \sigma_{\mathrm{c}}(\Delta c_{2,1}, \Delta c_{2,2}, \ldots \Delta c_{2,-n})) & \text{if} \quad n < 0 \end{cases}$$

Theorem 454 (Formula for $\epsilon_{\mathrm{c},n}(\Delta c)$) *If*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

is a pitch system and Δc is a chroma interval in ψ and n is an integer then

$$\epsilon_{\mathrm{c},n}(\Delta c) = (n \times \Delta c) \bmod \mu_{\mathrm{c}}$$

Proof

R1	Let		$n \in \mathbb{Z}$
R2	Let		$(1 \le k \le \mathrm{abs}(n)) \wedge (k \in \mathbb{Z})$
R3	Let		$\Delta c_{1,k} = \Delta c$ for all k
R4	Let		$\Delta c_{2,k} = \iota_{\mathrm{c}}(\Delta c)$ for all k
R5	R1 to R4 & 453	$\Rightarrow$	$\tau_{\mathrm{c}}(c, \epsilon_{\mathrm{c},n}(\Delta c)) = \begin{cases} \tau_{\mathrm{c}}(c, \sigma_{\mathrm{c}}(\Delta c_{1,1}, \Delta c_{1,2}, \ldots \Delta c_{1,n})) & \text{if} \quad n > 0 \\ c & \text{if} \quad n = 0 \\ \tau_{\mathrm{c}}(c, \sigma_{\mathrm{c}}(\Delta c_{2,1}, \Delta c_{2,2}, \ldots \Delta c_{2,-n})) & \text{if} \quad n < 0 \end{cases}$
R6	447	$\Rightarrow$	$\sigma_{\mathrm{c}}(\Delta c_{1,1}, \Delta c_{1,2}, \ldots \Delta c_{1,n}) = \left(\sum_{k=1}^{n} \Delta c_{1,k}\right) \bmod \mu_{\mathrm{c}}$
R7	R3 & R6	$\Rightarrow$	$\sigma_{\mathrm{c}}(\Delta c_{1,1}, \Delta c_{1,2}, \ldots \Delta c_{1,n}) = \left(\sum_{k=1}^{n} \Delta c\right) \bmod \mu_{\mathrm{c}} = (n \times \Delta c) \bmod \mu_{\mathrm{c}}$
R8	R5 & R7	$\Rightarrow$	$\tau_{\mathrm{c}}(c, \epsilon_{\mathrm{c},n}(\Delta c)) = \tau_{\mathrm{c}}(c, (n \times \Delta c) \bmod \mu_{\mathrm{c}})$ when $n > 0$
R9	407	$\Rightarrow$	$\tau_{\mathrm{c}}(c, (0 \times \Delta c) \bmod \mu_{\mathrm{c}}) = (c + 0) \bmod \mu_{\mathrm{c}} = c \bmod \mu_{\mathrm{c}}$
R10	72	$\Rightarrow$	$(0 \le c < \mu_{\mathrm{c}}) \wedge (c \in \mathbb{Z})$
R11	R9, R10 & 44	$\Rightarrow$	$\tau_{\mathrm{c}}(c, (n \times \Delta c) \bmod \mu_{\mathrm{c}}) = c$ when $n = 0$
R12	R5 & R11	$\Rightarrow$	$\tau_{\mathrm{c}}(c, \epsilon_{\mathrm{c},n}(\Delta c)) = \tau_{\mathrm{c}}(c, (n \times \Delta c) \bmod \mu_{\mathrm{c}})$ when $n = 0$
R13	447	$\Rightarrow$	$\sigma_{\mathrm{c}}(\Delta c_{2,1}, \Delta c_{2,2}, \ldots \Delta c_{2,-n}) = \left(\sum_{k=1}^{-n} \Delta c_{2,k}\right) \bmod \mu_{\mathrm{c}}$
R14	R4 & R13	$\Rightarrow$	$\sigma_{\mathrm{c}}(\Delta c_{2,1}, \Delta c_{2,2}, \ldots \Delta c_{2,-n}) = \left(\sum_{k=1}^{-n} \iota_{\mathrm{c}}(\Delta c)\right) \bmod \mu_{\mathrm{c}}$
			$= (-n \times \iota_{\mathrm{c}}(\Delta c)) \bmod \mu_{\mathrm{c}}$
R15	R14 & 451	$\Rightarrow$	$\sigma_{\mathrm{c}}(\Delta c_{2,1}, \Delta c_{2,2}, \ldots \Delta c_{2,-n}) = (-n \times ((-\Delta c) \bmod \mu_{\mathrm{c}})) \bmod \mu_{\mathrm{c}}$
R16	R15 & 45	$\Rightarrow$	$\sigma_{\mathrm{c}}(\Delta c_{2,1}, \Delta c_{2,2}, \ldots \Delta c_{2,-n}) = (-n \times (-\Delta c)) \bmod \mu_{\mathrm{c}}$
			$= (n \times \Delta c) \bmod \mu_{\mathrm{c}}$
R17	R5 & R16	$\Rightarrow$	$\tau_{\mathrm{c}}(c, \epsilon_{\mathrm{c},n}(\Delta c)) = \tau_{\mathrm{c}}(c, (n \times \Delta c) \bmod \mu_{\mathrm{c}})$ when $n < 0$
R18	R8, R12 & R17	$\Rightarrow$	$\tau_{\mathrm{c}}(c, \epsilon_{\mathrm{c},n}(\Delta c)) = \tau_{\mathrm{c}}(c, (n \times \Delta c) \bmod \mu_{\mathrm{c}})$ for all $n \in \mathbb{Z}$
R19	R18 & 410	$\Rightarrow$	$\epsilon_{\mathrm{c},n}(\Delta c) = (n \times \Delta c) \bmod \mu_{\mathrm{c}}$ for all $n \in \mathbb{Z}$

Theorem 455 *If*

$$\psi = [\mu_{\rm c}, \mu_{\rm m}, f_0, p_{{\rm c},0}]$$

is a pitch system and Δc *is any chroma interval in* ψ *then*

$$\iota_{\rm c}\left(\Delta c\right) = \epsilon_{{\rm c},-1}\left(\Delta c\right)$$

Proof

R1	454	$\Rightarrow$	$\epsilon_{{\rm c},-1}\left(\Delta c\right) = \left(-1 \times \Delta c\right) \bmod \mu_{\rm c}$
R2	451	$\Rightarrow$	$\iota_{\rm c}\left(\Delta c\right) = \left(-\Delta c\right) \bmod \mu_{\rm c}$
R3	R1 & R2	$\Rightarrow$	$\iota_{\rm c}\left(\Delta c\right) = \epsilon_{{\rm c},-1}\left(\Delta c\right)$

Theorem 456 *If*

$$\psi = [\mu_{\rm c}, \mu_{\rm m}, f_0, p_{{\rm c},0}]$$

is a pitch system, $n_1, n_2, \ldots n_k$ *is a collection of integers and* Δc *is a chroma interval in* ψ *then*

$$\epsilon_{{\rm c},n_k}\left(\ldots \epsilon_{{\rm c},n_2}\left(\epsilon_{{\rm c},n_1}\left(\Delta c\right)\right)\ldots\right) = \epsilon_{{\rm c},\prod_{j=1}^{k} n_j}\left(\Delta c\right)$$

Proof

R1			$\prod_{j=1}^{1} n_j = n_1$
R2	R1	$\Rightarrow$	$\epsilon_{\mathrm{c},n_1}(\Delta c) = \epsilon_{\mathrm{c},\prod_{j=1}^{1} n_j}(\Delta c)$
R3	R2	$\Rightarrow$	$\epsilon_{\mathrm{c},n_k}(\ldots\epsilon_{\mathrm{c},n_2}(\epsilon_{\mathrm{c},n_1}(\Delta c))\ldots) = \epsilon_{\mathrm{c},\prod_{j=1}^{k} n_j}(\Delta c)$ for $k=1$.
R4	453	$\Rightarrow$	$\left(\begin{array}{l}\epsilon_{\mathrm{c},n_k}(\ldots\epsilon_{\mathrm{c},n_2}(\epsilon_{\mathrm{c},n_1}(\Delta c))\ldots) = \epsilon_{\mathrm{c},\prod_{j=1}^{k} n_j}(\Delta c) \\ \Rightarrow \epsilon_{\mathrm{c},n_{k+1}}(\epsilon_{\mathrm{c},n_k}(\ldots\epsilon_{\mathrm{c},n_2}(\epsilon_{\mathrm{c},n_1}(\Delta c))\ldots)) = \epsilon_{\mathrm{c},n_{k+1}}\left(\epsilon_{\mathrm{c},\prod_{j=1}^{k} n_j}(\Delta c)\right)\end{array}\right)$
R5	454	$\Rightarrow$	$\begin{array}{l}\epsilon_{\mathrm{c},n_{k+1}}\left(\epsilon_{\mathrm{c},\prod_{j=1}^{k} n_j}(\Delta c)\right) \\ = \epsilon_{\mathrm{c},n_{k+1}}\left(\left(\prod_{j=1}^{k} n_j \times \Delta c\right) \bmod \mu_{\mathrm{c}}\right) \\ = \left(n_{k+1} \times \left(\left(\prod_{j=1}^{k} n_j \times \Delta c\right) \bmod \mu_{\mathrm{c}}\right)\right) \bmod \mu_{\mathrm{c}}\end{array}$
R6	R5 & 45	$\Rightarrow$	$\begin{array}{l}\epsilon_{\mathrm{c},n_{k+1}}\left(\epsilon_{\mathrm{c},\prod_{j=1}^{k} n_j}(\Delta c)\right) \\ = \left(n_{k+1} \times \prod_{j=1}^{k} n_j \times \Delta c\right) \bmod \mu_{\mathrm{c}} \\ = \left(\prod_{j=1}^{k+1} n_j \times \Delta c\right) \bmod \mu_{\mathrm{c}}\end{array}$
R7	454	$\Rightarrow$	$\epsilon_{\mathrm{c},\prod_{j=1}^{k+1} n_j}(\Delta c) = \left(\prod_{j=1}^{k+1} n_j \times \Delta c\right) \bmod \mu_{\mathrm{c}}$
R8	R6 & R7	$\Rightarrow$	$\epsilon_{\mathrm{c},\prod_{j=1}^{k+1} n_j}(\Delta c) = \epsilon_{\mathrm{c},n_{k+1}}\left(\epsilon_{\mathrm{c},\prod_{j=1}^{k} n_j}(\Delta c)\right)$
R9	R4 & R8	$\Rightarrow$	$\left(\begin{array}{l}\epsilon_{\mathrm{c},n_k}(\ldots\epsilon_{\mathrm{c},n_2}(\epsilon_{\mathrm{c},n_1}(\Delta c))\ldots) = \epsilon_{\mathrm{c},\prod_{j=1}^{k} n_j}(\Delta c) \\ \Rightarrow \epsilon_{\mathrm{c},n_{k+1}}(\epsilon_{\mathrm{c},n_k}(\ldots\epsilon_{\mathrm{c},n_2}(\epsilon_{\mathrm{c},n_1}(\Delta c))\ldots)) = \epsilon_{\mathrm{c},\prod_{j=1}^{k+1} n_j}(\Delta c)\end{array}\right)$
R10	R3 & R9	$\Rightarrow$	$\epsilon_{\mathrm{c},n_k}(\ldots\epsilon_{\mathrm{c},n_2}(\epsilon_{\mathrm{c},n_1}(\Delta c))\ldots) = \epsilon_{\mathrm{c},\prod_{j=1}^{k} n_j}(\Delta c)$ for all $k \in \mathbb{Z}, k > 0$.

Theorem 457 *If*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

is a pitch system, n is an integer and Δc is a chroma interval in ψ then

$$\iota_{\mathrm{c}}(\epsilon_{\mathrm{c},n}(\Delta c)) = \epsilon_{\mathrm{c},-n}(\Delta c)$$

Proof

R1	455	$\Rightarrow$	$\iota_{\mathrm{c}}(\Delta c) = \epsilon_{\mathrm{c},-1}(\Delta c)$
R2	R1	$\Rightarrow$	$\iota_{\mathrm{c}}(\epsilon_{\mathrm{c},n}(\Delta c)) = \epsilon_{\mathrm{c},-1}(\epsilon_{\mathrm{c},n}(\Delta c))$
R3	R2 & 456	$\Rightarrow$	$\iota_{\mathrm{c}}(\epsilon_{\mathrm{c},n}(\Delta c)) = \epsilon_{\mathrm{c},(-1\times n)}(\Delta c) = \epsilon_{\mathrm{c},-n}(\Delta c)$

Theorem 458 *If*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

is a pitch system, $n_1, n_2, \ldots n_k$ is a collection of integers and Δc is a chroma interval in ψ then

$$\sigma_{\mathrm{c}}\left(\epsilon_{\mathrm{c},n_1}(\Delta c), \epsilon_{\mathrm{c},n_2}(\Delta c), \ldots, \epsilon_{\mathrm{c},n_k}(\Delta c)\right) = \epsilon_{\mathrm{c},\sum_{j=1}^{k} n_j}(\Delta c)$$

Proof

R1	Let		$y = \sigma_{\mathrm{c}}\left(\epsilon_{\mathrm{c},n_1}(\Delta c), \epsilon_{\mathrm{c},n_2}(\Delta c), \ldots, \epsilon_{\mathrm{c},n_k}(\Delta c)\right)$
R2	R1 & 447	$\Rightarrow$	$y = \left(\sum_{j=1}^{k} \epsilon_{\mathrm{c},n_j}(\Delta c)\right) \bmod \mu_{\mathrm{c}}$
R3	R2 & 454	$\Rightarrow$	$y = \left(\sum_{j=1}^{k} ((n_j \times \Delta c) \bmod \mu_{\mathrm{c}})\right) \bmod \mu_{\mathrm{c}}$
R4	R3 & 39	$\Rightarrow$	$y = \left(\left(\sum_{j=1}^{k} n_j\right) \times \Delta c\right) \bmod \mu_{\mathrm{c}}$
R5	454	$\Rightarrow$	$\epsilon_{\mathrm{c},\sum_{j=1}^{k} n_j}(\Delta c) = \left(\left(\sum_{j=1}^{k} n_j\right) \times \Delta c\right) \bmod \mu_{\mathrm{c}}$
R6	R1, R4 & R5	$\Rightarrow$	$\sigma_{\mathrm{c}}\left(\epsilon_{\mathrm{c},n_1}(\Delta c), \epsilon_{\mathrm{c},n_2}(\Delta c), \ldots, \epsilon_{\mathrm{c},n_k}(\Delta c)\right) = \epsilon_{\mathrm{c},\sum_{j=1}^{k} n_j}(\Delta c)$

Exponentiation of the chroma tranposition function

Definition 459 (Definition of $\tau_{\mathrm{c},n}(c, \Delta c)$) *If ψ is a pitch system and c is a chroma in ψ and Δc is a chroma interval in ψ then*

$$\tau_{\mathrm{c},n}(c, \Delta c) = \tau_{\mathrm{c}}\left(c, \epsilon_{\mathrm{c},n}(\Delta c)\right)$$

Theorem 460 *If*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

is a pitch system, $n_1, n_2, \ldots, n_k$ is a collection of integers, c is a chroma in ψ and Δc is a chroma interval in ψ then

$$\tau_{\mathrm{c},n_k}\left(\ldots \tau_{\mathrm{c},n_2}\left(\tau_{\mathrm{c},n_1}(c, \Delta c), \Delta c\right) \ldots, \Delta c\right) = \tau_{\mathrm{c},\sum_{j=1}^{k} n_j}(c, \Delta c)$$

Proof

R1	Let		$z = \tau_{\mathrm{c},n_k}\left(\ldots\tau_{\mathrm{c},n_2}\left(\tau_{\mathrm{c},n_1}\left(c,\Delta c\right),\Delta c\right)\ldots,\Delta c\right)$
R2	Let		$y = \tau_{\mathrm{c},\sum_{j=1}^{k} n_j}\left(c,\Delta c\right)$
R3	R1 & 459	$\Rightarrow$	$z = \tau_{\mathrm{c}}\left(\ldots\tau_{\mathrm{c}}\left(\tau_{\mathrm{c}}\left(c,\epsilon_{\mathrm{c},n_1}\left(\Delta c\right)\right),\epsilon_{\mathrm{c},n_2}\left(\Delta c\right)\right)\ldots,\epsilon_{\mathrm{c},n_k}\left(\Delta c\right)\right)$
R4	R3 & 454	$\Rightarrow$	$z = \tau_{\mathrm{c}}\left(\ldots\tau_{\mathrm{c}}\left(\tau_{\mathrm{c}}\left(c,\left(n_1\times\Delta c\right)\bmod\mu_{\mathrm{c}}\right),\left(n_2\times\Delta c\right)\bmod\mu_{\mathrm{c}}\right)\ldots,\left(n_k\times\Delta c\right)\bmod\mu_{\mathrm{c}}\right)$
R5	R4 & 407	$\Rightarrow$	$z = \left(\begin{array}{l}\ldots((c+(n_1\times\Delta c)\bmod\mu_{\mathrm{c}})\bmod\mu_{\mathrm{c}}+(n_2\times\Delta c)\bmod\mu_{\mathrm{c}})\bmod\mu_{\mathrm{c}}\ldots\\ +(n_k\times\Delta c)\bmod\mu_{\mathrm{c}}\end{array}\right)\bmod\mu_{\mathrm{c}}$
R6	R5 & 38	$\Rightarrow$	$z = (c+n_1\times\Delta c+n_2\times\Delta c+\ldots+n_k\times\Delta c)\bmod\mu_{\mathrm{c}}$
			$= (c+(n_1+n_2+\ldots+n_k)\times\Delta c)\bmod\mu_{\mathrm{c}}$
			$= \left(c+\left(\sum_{j=1}^{k} n_j\right)\times\Delta c\right)\bmod\mu_{\mathrm{c}}$
R7	R2 & 459	$\Rightarrow$	$y = \tau_{\mathrm{c}}\left(c,\epsilon_{\mathrm{c},\sum_{j=1}^{k} n_j}\left(\Delta c\right)\right)$
R8	R7 & 407	$\Rightarrow$	$y = \left(c+\epsilon_{\mathrm{c},\sum_{j=1}^{k} n_j}\left(\Delta c\right)\right)\bmod\mu_{\mathrm{c}}$
R9	R8 & 454	$\Rightarrow$	$y = \left(c+\left(\left(\sum_{j=1}^{k} n_j\right)\times\Delta c\right)\bmod\mu_{\mathrm{c}}\right)\bmod\mu_{\mathrm{c}}$
R10	R9 & 38	$\Rightarrow$	$y = \left(c+\left(\sum_{j=1}^{k} n_j\right)\times\Delta c\right)\bmod\mu_{\mathrm{c}}$
R11	R6 & R10	$\Rightarrow$	$y = z$
R12	R1, R2 & R11	$\Rightarrow$	$\tau_{\mathrm{c},n_k}\left(\ldots\tau_{\mathrm{c},n_2}\left(\tau_{\mathrm{c},n_1}\left(c,\Delta c\right),\Delta c\right)\ldots,\Delta c\right) = \tau_{\mathrm{c},\sum_{j=1}^{k} n_j}\left(c,\Delta c\right)$

4.6.2 Summation, inversion and exponentiation of morph intervals

Summation of morph intervals

Definition 461 (Definition of $\sigma_{\mathrm{m}}\left(\Delta m_1,\Delta m_2,\ldots,\Delta m_n\right)$) *If*

$$\psi = [\mu_{\mathrm{c}},\mu_{\mathrm{m}},f_0,p_{\mathrm{c},0}]$$

is a pitch system and

$$\Delta m_1,\Delta m_2,\ldots,\Delta m_n$$

is a collection of morph intervals in ψ then

$$\sigma_{\mathrm{m}}\left(\Delta m_1,\Delta m_2,\ldots,\Delta m_n\right) = \left(\sum_{k=1}^{n}\Delta m_k\right)\bmod\mu_{\mathrm{m}}$$

Theorem 462 *If ψ is a pitch system and*

$$\Delta m_1, \Delta m_2, \ldots, \Delta m_n$$

is a collection of morph intervals in ψ and m is a morph in ψ then

$$\tau_{\mathrm{m}}\left(m, \sigma_{\mathrm{m}}\left(\Delta m_1, \Delta m_2, \ldots, \Delta m_n\right)\right) = \tau_{\mathrm{m}}\left(\ldots \tau_{\mathrm{m}}\left(\tau_{\mathrm{m}}\left(m, \Delta m_1\right), \Delta m_2\right) \ldots, \Delta m_n\right)$$

Proof

R1 412 $\Rightarrow$ $\tau_{\mathrm{m}}\left(\ldots \tau_{\mathrm{m}}\left(\tau_{\mathrm{m}}\left(m, \Delta m_1\right), \Delta m_2\right) \ldots, \Delta m_n\right)$

$= \tau_{\mathrm{m}}\left(\ldots \tau_{\mathrm{m}}\left(\left(m + \Delta m_1\right) \bmod \mu_{\mathrm{m}}, \Delta m_2\right) \ldots, \Delta m_n\right)$

$= \left(\ldots\left(\left(m + \Delta m_1\right) \bmod \mu_{\mathrm{m}} + \Delta m_2\right) \bmod \mu_{\mathrm{m}} \ldots + \Delta m_n\right) \bmod \mu_{\mathrm{m}}$

R2 R1 & 38 $\Rightarrow$ $\tau_{\mathrm{m}}\left(\ldots \tau_{\mathrm{m}}\left(\tau_{\mathrm{m}}\left(m, \Delta m_1\right), \Delta m_2\right) \ldots, \Delta m_n\right)$

$= \left(m + \Delta m_1 + \Delta m_2 + \ldots + \Delta m_n\right) \bmod \mu_{\mathrm{m}}$

$= \left(m + \sum_{k=1}^{n} \Delta m_k\right) \bmod \mu_{\mathrm{m}}$

R3 R2 & 38 $\Rightarrow$ $\tau_{\mathrm{m}}\left(\ldots \tau_{\mathrm{m}}\left(\tau_{\mathrm{m}}\left(m, \Delta m_1\right), \Delta m_2\right) \ldots, \Delta m_n\right)$

$= \left(m + \left(\sum_{k=1}^{n} \Delta m_k\right) \bmod \mu_{\mathrm{m}}\right) \bmod \mu_{\mathrm{m}}$

R4 R3 & 461 $\Rightarrow$ $\tau_{\mathrm{m}}\left(\ldots \tau_{\mathrm{m}}\left(\tau_{\mathrm{m}}\left(m, \Delta m_1\right), \Delta m_2\right) \ldots, \Delta m_n\right)$

$= \left(m + \sigma_{\mathrm{m}}\left(\Delta m_1, \Delta m_2, \ldots, \Delta m_n\right)\right) \bmod \mu_{\mathrm{m}}$

R5 R4 & 412 $\Rightarrow$ $\tau_{\mathrm{m}}\left(\ldots \tau_{\mathrm{m}}\left(\tau_{\mathrm{m}}\left(m, \Delta m_1\right), \Delta m_2\right) \ldots, \Delta m_n\right)$

$= \tau_{\mathrm{m}}\left(m, \sigma_{\mathrm{m}}\left(\Delta m_1, \Delta m_2, \ldots, \Delta m_n\right)\right)$

Inversion of morph intervals

Definition 463 (Definition of $\iota_{\mathrm{m}}\left(\Delta m\right)$) *If ψ is a pitch system and Δm is a morph interval in ψ and m is a morph in ψ then $\iota_{\mathrm{m}}\left(\Delta m\right)$ is the morph interval that satisfies the following equation*

$$\tau_{\mathrm{m}}\left(\tau_{\mathrm{m}}\left(m, \Delta m\right), \iota_{\mathrm{m}}\left(\Delta m\right)\right) = m$$

Definition 464 (Inversional equivalence of morph intervals) *If ψ is a pitch system and Δm_1 and Δm_2 are morph intervals in ψ then Δm_1 and Δm_2 are* inversionally equivalent *if and only if*

$$\left(\iota_{\mathrm{m}}\left(\Delta m_1\right) = \Delta m_2\right) \vee \left(\Delta m_1 = \Delta m_2\right)$$

The fact that two morph intervals are inversionally equivalent is denoted as follows:

$$\Delta m_1 \equiv_{\iota} \Delta m_2$$

Theorem 465 *If*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

is a pitch system and Δm *is a morph interval in* ψ *and* m *is a morph in* ψ *then*

$$\iota_{\mathrm{m}}\left(\Delta m\right) = \left(-\Delta m\right) \bmod \mu_{\mathrm{m}}$$

Proof

R1	463	$\Rightarrow$	$\tau_{\mathrm{m}}\left(\tau_{\mathrm{m}}\left(m, \Delta m\right), \iota_{\mathrm{m}}\left(\Delta m\right)\right) = m$
R2	412	$\Rightarrow$	$\tau_{\mathrm{m}}\left(\tau_{\mathrm{m}}\left(m, \Delta m\right), \left(-\Delta m\right) \bmod \mu_{\mathrm{m}}\right)$
			$= \tau_{\mathrm{m}}\left(\left(m + \Delta m\right) \bmod \mu_{\mathrm{m}}, \left(-\Delta m\right) \bmod \mu_{\mathrm{m}}\right)$
			$= \left(\left(m + \Delta m\right) \bmod \mu_{\mathrm{m}} + \left(-\Delta m\right) \bmod \mu_{\mathrm{m}}\right) \bmod \mu_{\mathrm{m}}$
R3	R2 &34	$\Rightarrow$	$\tau_{\mathrm{m}}\left(\tau_{\mathrm{m}}\left(m, \Delta m\right), \left(-\Delta m\right) \bmod \mu_{\mathrm{m}}\right)$
			$= \left(m + \Delta m - \Delta m\right) \bmod \mu_{\mathrm{m}}$
			$= m \bmod \mu_{\mathrm{m}}$
R4	77	$\Rightarrow$	$\left(0 \leq m < \mu_{\mathrm{m}}\right) \wedge \left(m \in \mathbb{Z}\right)$
R5	R3, R4 & 44	$\Rightarrow$	$\tau_{\mathrm{m}}\left(\tau_{\mathrm{m}}\left(m, \Delta m\right), \left(-\Delta m\right) \bmod \mu_{\mathrm{m}}\right) = m$
R6	R5 & R1	$\Rightarrow$	$\tau_{\mathrm{m}}\left(\tau_{\mathrm{m}}\left(m, \Delta m\right), \left(-\Delta m\right) \bmod \mu_{\mathrm{m}}\right) = \tau_{\mathrm{m}}\left(\tau_{\mathrm{m}}\left(m, \Delta m\right), \iota_{\mathrm{m}}\left(\Delta m\right)\right)$
R7	R6 & 415	$\Rightarrow$	$\iota_{\mathrm{m}}\left(\Delta m\right) = \left(-\Delta m\right) \bmod \mu_{\mathrm{m}}$

Theorem 466 *If* ψ *is a pitch system and* Δm*,* Δm_1 *and* Δm_2 *are morph intervals in* ψ *then*

$$\left(\Delta m_1 = \iota_{\mathrm{m}}\left(\Delta m\right)\right) \wedge \left(\Delta m_2 = \iota_{\mathrm{m}}\left(\Delta m\right)\right) \Rightarrow \left(\Delta m_1 = \Delta m_2\right)$$

Proof

R1	Let		$\Delta m_1 = \iota_{\mathrm{m}}(\Delta m)$
R2	Let		$\Delta m_2 = \iota_{\mathrm{m}}(\Delta m)$
R3	R1 & 463	$\Rightarrow$	$\tau_{\mathrm{m}}(\tau_{\mathrm{m}}(m, \Delta m), \Delta m_1) = m$
R4	R2 & 463	$\Rightarrow$	$\tau_{\mathrm{m}}(\tau_{\mathrm{m}}(m, \Delta m), \Delta m_2) = m$
R5	R3 & R4	$\Rightarrow$	$\tau_{\mathrm{m}}(\tau_{\mathrm{m}}(m, \Delta m), \Delta m_1) = \tau_{\mathrm{m}}(\tau_{\mathrm{m}}(m, \Delta m), \Delta m_2)$
R6	R5 & 415	$\Rightarrow$	$\Delta m_1 = \Delta m_2$
R7	R1 to R6	$\Rightarrow$	$(\Delta m_1 = \iota_{\mathrm{m}}(\Delta m)) \wedge (\Delta m_2 = \iota_{\mathrm{m}}(\Delta m)) \Rightarrow (\Delta m_1 = \Delta m_2)$

Exponentiation of morph intervals

Definition 467 (Definition of $\epsilon_{\mathrm{m},n}(\Delta m)$) *Given that:*

1. *ψ is a pitch system;*
2. *m is a morph in ψ;*
3. *Δm is a morph interval in ψ;*
4. *n is an integer;*
5. *k is an integer and $1 \leq k \leq \mathrm{abs}(n)$;*
6. *$\Delta m_{1,k} = \Delta m$ for all k; and*
7. *$\Delta m_{2,k} = \iota_{\mathrm{m}}(\Delta m)$ for all k;*

then $\epsilon_{\mathrm{m},n}(\Delta m)$ is any morph interval that satisfies the following equation:

$$\tau_{\mathrm{m}}(m, \epsilon_{\mathrm{m},n}(\Delta m)) = \begin{cases} \tau_{\mathrm{m}}(m, \sigma_{\mathrm{m}}(\Delta m_{1,1}, \Delta m_{1,2}, \ldots \Delta m_{1,n})) & \text{if} \quad n > 0 \\ m & \text{if} \quad n = 0 \\ \tau_{\mathrm{m}}(m, \sigma_{\mathrm{m}}(\Delta m_{2,1}, \Delta m_{2,2}, \ldots \Delta m_{2,-n})) & \text{if} \quad n < 0 \end{cases}$$

Theorem 468 (Formula for $\epsilon_{\mathrm{m},n}(\Delta m)$) *If*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

is a pitch system and Δm is a morph interval in ψ and n is an integer then

$$\epsilon_{\mathrm{m},n}(\Delta m) = (n \times \Delta m) \bmod \mu_{\mathrm{m}}$$

Proof

R1	Let		$n \in \mathbb{Z}$
R2	Let		$(1 \leq k \leq \mathrm{abs}\,(n)) \wedge (k \in \mathbb{Z})$
R3	Let		$\Delta m_{1,k} = \Delta m$ for all k
R4	Let		$\Delta m_{2,k} = \iota_{\mathrm{m}}\,(\Delta m)$ for all k
R5	R1 to R4 & 467	$\Rightarrow$	$\tau_{\mathrm{m}}\,(m, \epsilon_{\mathrm{m},n}\,(\Delta m)) = \begin{cases} \tau_{\mathrm{m}}\,(m, \sigma_{\mathrm{m}}\,(\Delta m_{1,1}, \Delta m_{1,2}, \ldots \Delta m_{1,n})) & \text{if} \quad n > 0 \\ m & \text{if} \quad n = 0 \\ \tau_{\mathrm{m}}\,(m, \sigma_{\mathrm{m}}\,(\Delta m_{2,1}, \Delta m_{2,2}, \ldots \Delta m_{2,-n})) & \text{if} \quad n < 0 \end{cases}$
R6	461	$\Rightarrow$	$\sigma_{\mathrm{m}}\,(\Delta m_{1,1}, \Delta m_{1,2}, \ldots \Delta m_{1,n}) = \left(\sum_{k=1}^{n} \Delta m_{1,k}\right) \bmod \mu_{\mathrm{m}}$
R7	R3 & R6	$\Rightarrow$	$\sigma_{\mathrm{m}}\,(\Delta m_{1,1}, \Delta m_{1,2}, \ldots \Delta m_{1,n}) = \left(\sum_{k=1}^{n} \Delta m\right) \bmod \mu_{\mathrm{m}} = (n \times \Delta m) \bmod \mu_{\mathrm{m}}$
R8	R5 & R7	$\Rightarrow$	$\tau_{\mathrm{m}}\,(m, \epsilon_{\mathrm{m},n}\,(\Delta m)) = \tau_{\mathrm{m}}\,(m, (n \times \Delta m) \bmod \mu_{\mathrm{m}})$ when $n > 0$
R9	412	$\Rightarrow$	$\tau_{\mathrm{m}}\,(m, (0 \times \Delta m) \bmod \mu_{\mathrm{m}}) = (m + 0) \bmod \mu_{\mathrm{m}} = m \bmod \mu_{\mathrm{m}}$
R10	77	$\Rightarrow$	$(0 \leq m < \mu_{\mathrm{m}}) \wedge (m \in \mathbb{Z})$
R11	R9, R10 & 44	$\Rightarrow$	$\tau_{\mathrm{m}}\,(m, (n \times \Delta m) \bmod \mu_{\mathrm{m}}) = m$ when $n = 0$
R12	R5 & R11	$\Rightarrow$	$\tau_{\mathrm{m}}\,(m, \epsilon_{\mathrm{m},n}\,(\Delta m)) = \tau_{\mathrm{m}}\,(m, (n \times \Delta m) \bmod \mu_{\mathrm{m}})$ when $n = 0$
R13	461	$\Rightarrow$	$\sigma_{\mathrm{m}}\,(\Delta m_{2,1}, \Delta m_{2,2}, \ldots \Delta m_{2,-n}) = \left(\sum_{k=1}^{-n} \Delta m_{2,k}\right) \bmod \mu_{\mathrm{m}}$
R14	R4 & R13	$\Rightarrow$	$\sigma_{\mathrm{m}}\,(\Delta m_{2,1}, \Delta m_{2,2}, \ldots \Delta m_{2,-n}) = \left(\sum_{k=1}^{-n} \iota_{\mathrm{m}}\,(\Delta m)\right) \bmod \mu_{\mathrm{m}}$
			$= (-n \times \iota_{\mathrm{m}}\,(\Delta m)) \bmod \mu_{\mathrm{m}}$
R15	R14 & 465	$\Rightarrow$	$\sigma_{\mathrm{m}}\,(\Delta m_{2,1}, \Delta m_{2,2}, \ldots \Delta m_{2,-n}) = (-n \times ((-\Delta m) \bmod \mu_{\mathrm{m}})) \bmod \mu_{\mathrm{m}}$
R16	R15 & 45	$\Rightarrow$	$\sigma_{\mathrm{m}}\,(\Delta m_{2,1}, \Delta m_{2,2}, \ldots \Delta m_{2,-n}) = (-n \times (-\Delta m)) \bmod \mu_{\mathrm{m}}$
			$= (n \times \Delta m) \bmod \mu_{\mathrm{m}}$
R17	R5 & R16	$\Rightarrow$	$\tau_{\mathrm{m}}\,(m, \epsilon_{\mathrm{m},n}\,(\Delta m)) = \tau_{\mathrm{m}}\,(m, (n \times \Delta m) \bmod \mu_{\mathrm{m}})$ when $n < 0$
R18	R8, R12 & R17	$\Rightarrow$	$\tau_{\mathrm{m}}\,(m, \epsilon_{\mathrm{m},n}\,(\Delta m)) = \tau_{\mathrm{m}}\,(m, (n \times \Delta m) \bmod \mu_{\mathrm{m}})$ for all $n \in \mathbb{Z}$
R19	R18 & 415	$\Rightarrow$	$\epsilon_{\mathrm{m},n}\,(\Delta m) = (n \times \Delta m) \bmod \mu_{\mathrm{m}}$ for all $n \in \mathbb{Z}$

Theorem 469 *If*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

is a pitch system and Δm *is any morph interval in* ψ *then*

$$\iota_{\mathrm{m}}(\Delta m) = \epsilon_{\mathrm{m},-1}(\Delta m)$$

Proof

R1	468	$\Rightarrow$	$\epsilon_{\mathrm{m},-1}(\Delta m) = (-1 \times \Delta m) \bmod \mu_{\mathrm{m}}$
R2	465	$\Rightarrow$	$\iota_{\mathrm{m}}(\Delta m) = (-\Delta m) \bmod \mu_{\mathrm{m}}$
R3	R1 & R2	$\Rightarrow$	$\iota_{\mathrm{m}}(\Delta m) = \epsilon_{\mathrm{m},-1}(\Delta m)$

Theorem 470 *If*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

is a pitch system, $n_1, n_2, \ldots n_k$ *is a collection of integers and* Δm *is a morph interval in* ψ *then*

$$\epsilon_{\mathrm{m},n_k}(\ldots \epsilon_{\mathrm{m},n_2}(\epsilon_{\mathrm{m},n_1}(\Delta m)) \ldots) = \epsilon_{\mathrm{m},\prod_{j=1}^{k} n_j}(\Delta m)$$

Proof

R1			$\prod_{j=1}^{1} n_j = n_1$
R2	R1	$\Rightarrow$	$\epsilon_{\mathrm{m},n_1}(\Delta m) = \epsilon_{\mathrm{m},\prod_{j=1}^{1} n_j}(\Delta m)$
R3	R2	$\Rightarrow$	$\epsilon_{\mathrm{m},n_k}(\ldots \epsilon_{\mathrm{m},n_2}(\epsilon_{\mathrm{m},n_1}(\Delta m))\ldots) = \epsilon_{\mathrm{m},\prod_{j=1}^{k} n_j}(\Delta m)$ for $k = 1$.
R4	467	$\Rightarrow$	$\left(\begin{array}{l}\epsilon_{\mathrm{m},n_k}(\ldots \epsilon_{\mathrm{m},n_2}(\epsilon_{\mathrm{m},n_1}(\Delta m))\ldots) = \epsilon_{\mathrm{m},\prod_{j=1}^{k} n_j}(\Delta m) \\ \Rightarrow \epsilon_{\mathrm{m},n_{k+1}}(\epsilon_{\mathrm{m},n_k}(\ldots \epsilon_{\mathrm{m},n_2}(\epsilon_{\mathrm{m},n_1}(\Delta m))\ldots)) = \epsilon_{\mathrm{m},n_{k+1}}\left(\epsilon_{\mathrm{m},\prod_{j=1}^{k} n_j}(\Delta m)\right)\end{array}\right)$
R5	468	$\Rightarrow$	$\begin{array}{l}\epsilon_{\mathrm{m},n_{k+1}}\left(\epsilon_{\mathrm{m},\prod_{j=1}^{k} n_j}(\Delta m)\right) \\ = \epsilon_{\mathrm{m},n_{k+1}}\left(\left(\prod_{j=1}^{k} n_j \times \Delta m\right) \bmod \mu_\mathrm{m}\right) \\ = \left(n_{k+1} \times \left(\left(\prod_{j=1}^{k} n_j \times \Delta m\right) \bmod \mu_\mathrm{m}\right)\right) \bmod \mu_\mathrm{m}\end{array}$
R6	R5 & 45	$\Rightarrow$	$\begin{array}{l}\epsilon_{\mathrm{m},n_{k+1}}\left(\epsilon_{\mathrm{m},\prod_{j=1}^{k} n_j}(\Delta m)\right) \\ = \left(n_{k+1} \times \prod_{j=1}^{k} n_j \times \Delta m\right) \bmod \mu_\mathrm{m} \\ = \left(\prod_{j=1}^{k+1} n_j \times \Delta m\right) \bmod \mu_\mathrm{m}\end{array}$
R7	468	$\Rightarrow$	$\epsilon_{\mathrm{m},\prod_{j=1}^{k+1} n_j}(\Delta m) = \left(\prod_{j=1}^{k+1} n_j \times \Delta m\right) \bmod \mu_\mathrm{m}$
R8	R6 & R7	$\Rightarrow$	$\epsilon_{\mathrm{m},\prod_{j=1}^{k+1} n_j}(\Delta m) = \epsilon_{\mathrm{m},n_{k+1}}\left(\epsilon_{\mathrm{m},\prod_{j=1}^{k} n_j}(\Delta m)\right)$
R9	R4 & R8	$\Rightarrow$	$\left(\begin{array}{l}\epsilon_{\mathrm{m},n_k}(\ldots \epsilon_{\mathrm{m},n_2}(\epsilon_{\mathrm{m},n_1}(\Delta m))\ldots) = \epsilon_{\mathrm{m},\prod_{j=1}^{k} n_j}(\Delta m) \\ \Rightarrow \epsilon_{\mathrm{m},n_{k+1}}(\epsilon_{\mathrm{m},n_k}(\ldots \epsilon_{\mathrm{m},n_2}(\epsilon_{\mathrm{m},n_1}(\Delta m))\ldots)) = \epsilon_{\mathrm{m},\prod_{j=1}^{k+1} n_j}(\Delta m)\end{array}\right)$
R10	R3 & R9	$\Rightarrow$	$\epsilon_{\mathrm{m},n_k}(\ldots \epsilon_{\mathrm{m},n_2}(\epsilon_{\mathrm{m},n_1}(\Delta m))\ldots) = \epsilon_{\mathrm{m},\prod_{j=1}^{k} n_j}(\Delta m)$ for all $k \in \mathbb{Z}, k > 0$.

Theorem 471 *If*

$$\psi = [\mu_\mathrm{c}, \mu_\mathrm{m}, f_0, p_{\mathrm{c},0}]$$

is a pitch system, n is an integer and Δm is a morph interval in ψ then

$$\iota_\mathrm{m}(\epsilon_{\mathrm{m},n}(\Delta m)) = \epsilon_{\mathrm{m},-n}(\Delta m)$$

Proof

R1	469	$\Rightarrow$	$\iota_\mathrm{m}(\Delta m) = \epsilon_{\mathrm{m},-1}(\Delta m)$
R2	R1	$\Rightarrow$	$\iota_\mathrm{m}(\epsilon_{\mathrm{m},n}(\Delta m)) = \epsilon_{\mathrm{m},-1}(\epsilon_{\mathrm{m},n}(\Delta m))$
R3	R2 & 470	$\Rightarrow$	$\iota_\mathrm{m}(\epsilon_{\mathrm{m},n}(\Delta m)) = \epsilon_{\mathrm{m},(-1\times n)}(\Delta m) = \epsilon_{\mathrm{m},-n}(\Delta m)$

Theorem 472 *If*

$$\psi = [\mu_\mathrm{c}, \mu_\mathrm{m}, f_0, p_{\mathrm{c},0}]$$

is a pitch system, $n_1, n_2, \ldots n_k$ is a collection of integers and Δm is a morph interval in ψ then

$$\sigma_{\mathrm{m}}\left(\epsilon_{\mathrm{m},n_1}(\Delta m), \epsilon_{\mathrm{m},n_2}(\Delta m), \ldots, \epsilon_{\mathrm{m},n_k}(\Delta m)\right) = \epsilon_{\mathrm{m},\sum_{j=1}^{k} n_j}(\Delta m)$$

Proof

R1	Let		$y = \sigma_{\mathrm{m}}\left(\epsilon_{\mathrm{m},n_1}(\Delta m), \epsilon_{\mathrm{m},n_2}(\Delta m), \ldots, \epsilon_{\mathrm{m},n_k}(\Delta m)\right)$
R2	R1 & 461	$\Rightarrow$	$y = \left(\sum_{j=1}^{k} \epsilon_{\mathrm{m},n_j}(\Delta m)\right) \bmod \mu_{\mathrm{m}}$
R3	R2 & 468	$\Rightarrow$	$y = \left(\sum_{j=1}^{k} ((n_j \times \Delta m) \bmod \mu_{\mathrm{m}})\right) \bmod \mu_{\mathrm{m}}$
R4	R3 & 39	$\Rightarrow$	$y = \left(\left(\sum_{j=1}^{k} n_j\right) \times \Delta m\right) \bmod \mu_{\mathrm{m}}$
R5	468	$\Rightarrow$	$\epsilon_{\mathrm{m},\sum_{j=1}^{k} n_j}(\Delta m) = \left(\left(\sum_{j=1}^{k} n_j\right) \times \Delta m\right) \bmod \mu_{\mathrm{m}}$
R6	R1, R4 & R5	$\Rightarrow$	$\sigma_{\mathrm{m}}\left(\epsilon_{\mathrm{m},n_1}(\Delta m), \epsilon_{\mathrm{m},n_2}(\Delta m), \ldots, \epsilon_{\mathrm{m},n_k}(\Delta m)\right) = \epsilon_{\mathrm{m},\sum_{j=1}^{k} n_j}(\Delta m)$

Exponentiation of the morph tranposition function

Definition 473 (Definition of $\tau_{\mathrm{m},n}(m, \Delta m)$) *If ψ is a pitch system and m is a morph in ψ and Δm is a morph interval in ψ then*

$$\tau_{\mathrm{m},n}(m, \Delta m) = \tau_{\mathrm{m}}\left(m, \epsilon_{\mathrm{m},n}(\Delta m)\right)$$

Theorem 474 *If*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

is a pitch system, $n_1, n_2, \ldots, n_k$ is a collection of integers, m is a morph in ψ and Δm is a morph interval in ψ then

$$\tau_{\mathrm{m},n_k}\left(\ldots \tau_{\mathrm{m},n_2}\left(\tau_{\mathrm{m},n_1}(m, \Delta m), \Delta m\right) \ldots, \Delta m\right) = \tau_{\mathrm{m},\sum_{j=1}^{k} n_j}(m, \Delta m)$$

Proof

R1	Let		$z = \tau_{\mathrm{m},n_k}\left(\ldots\tau_{\mathrm{m},n_2}\left(\tau_{\mathrm{m},n_1}\left(m,\Delta m\right),\Delta m\right)\ldots,\Delta m\right)$
R2	Let		$y = \tau_{\mathrm{m},\sum_{j=1}^{k} n_j}\left(m,\Delta m\right)$
R3	R1 & 473	$\Rightarrow$	$z = \tau_{\mathrm{m}}\left(\ldots\tau_{\mathrm{m}}\left(\tau_{\mathrm{m}}\left(m,\epsilon_{\mathrm{m},n_1}\left(\Delta m\right)\right),\epsilon_{\mathrm{m},n_2}\left(\Delta m\right)\right)\ldots,\epsilon_{\mathrm{m},n_k}\left(\Delta m\right)\right)$
R4	R3 & 468	$\Rightarrow$	$z = \tau_{\mathrm{m}}\left(\begin{array}{l}\ldots\tau_{\mathrm{m}}\left(\tau_{\mathrm{m}}\left(m,\left(n_1\times\Delta m\right)\bmod\mu_{\mathrm{m}}\right),\left(n_2\times\Delta m\right)\bmod\mu_{\mathrm{m}}\right)\ldots,\\ \left(n_k\times\Delta m\right)\bmod\mu_{\mathrm{m}}\end{array}\right)$
R5	R4 & 412	$\Rightarrow$	$z = \left(\begin{array}{l}\ldots\left(\begin{array}{l}\left(m+\left(n_1\times\Delta m\right)\bmod\mu_{\mathrm{m}}\right)\bmod\mu_{\mathrm{m}}\\ +\left(n_2\times\Delta m\right)\bmod\mu_{\mathrm{m}}\end{array}\right)\bmod\mu_{\mathrm{m}}\ldots\\ +\left(n_k\times\Delta m\right)\bmod\mu_{\mathrm{m}}\end{array}\right)\bmod\mu_{\mathrm{m}}$
R6	R5 & 38	$\Rightarrow$	$z = \left(m+n_1\times\Delta m+n_2\times\Delta m+\ldots+n_k\times\Delta m\right)\bmod\mu_{\mathrm{m}}$
			$= \left(m+\left(n_1+n_2+\ldots+n_k\right)\times\Delta m\right)\bmod\mu_{\mathrm{m}}$
			$= \left(m+\left(\sum_{j=1}^{k} n_j\right)\times\Delta m\right)\bmod\mu_{\mathrm{m}}$
R7	R2 & 473	$\Rightarrow$	$y = \tau_{\mathrm{m}}\left(m,\epsilon_{\mathrm{m},\sum_{j=1}^{k} n_j}\left(\Delta m\right)\right)$
R8	R7 & 412	$\Rightarrow$	$y = \left(m+\epsilon_{\mathrm{m},\sum_{j=1}^{k} n_j}\left(\Delta m\right)\right)\bmod\mu_{\mathrm{m}}$
R9	R8 & 468	$\Rightarrow$	$y = \left(m+\left(\left(\sum_{j=1}^{k} n_j\right)\times\Delta m\right)\bmod\mu_{\mathrm{m}}\right)\bmod\mu_{\mathrm{m}}$
R10	R9 & 38	$\Rightarrow$	$y = \left(m+\left(\sum_{j=1}^{k} n_j\right)\times\Delta m\right)\bmod\mu_{\mathrm{m}}$
R11	R6 & R10	$\Rightarrow$	$y = z$
R12	R1, R2 & R11	$\Rightarrow$	$\tau_{\mathrm{m},n_k}\left(\ldots\tau_{\mathrm{m},n_2}\left(\tau_{\mathrm{m},n_1}\left(m,\Delta m\right),\Delta m\right)\ldots,\Delta m\right) = \tau_{\mathrm{m},\sum_{j=1}^{k} n_j}\left(m,\Delta m\right)$

4.6.3 Summation, inversion and exponentiation of chromamorph intervals

Summation of chromamorph intervals

Definition 475 (Definition of $\sigma_{\mathrm{q}}\left(\Delta q_1,\Delta q_2,\ldots,\Delta q_n\right)$) *If*

$$\psi = [\mu_{\mathrm{c}},\mu_{\mathrm{m}},f_0,p_{\mathrm{c},0}]$$

is a pitch system and

$$\Delta q_1,\Delta q_2,\ldots,\Delta q_n$$

is a collection of chromamorph intervals in ψ then

$$\sigma_{\mathrm{q}}\left(\Delta q_1, \Delta q_2, \ldots, \Delta q_n\right) = \left[\begin{array}{l} \sigma_{\mathrm{c}}\left(\Delta\,\mathrm{c}\left(\Delta q_1\right), \Delta\,\mathrm{c}\left(\Delta q_2\right), \ldots \Delta\,\mathrm{c}\left(\Delta q_n\right)\right), \\ \sigma_{\mathrm{m}}\left(\Delta\,\mathrm{m}\left(\Delta q_1\right), \Delta\,\mathrm{m}\left(\Delta q_2\right), \ldots \Delta\,\mathrm{m}\left(\Delta q_n\right)\right) \end{array}\right]$$

Theorem 476 *If ψ is a pitch system and*

$$\Delta q_1, \Delta q_2, \ldots, \Delta q_n$$

is a collection of chromamorph intervals in ψ and q is a chromamorph in ψ then

$$\tau_{\mathrm{q}}\left(q, \sigma_{\mathrm{q}}\left(\Delta q_1, \Delta q_2, \ldots, \Delta q_n\right)\right) = \tau_{\mathrm{q}}\left(\ldots \tau_{\mathrm{q}}\left(\tau_{\mathrm{q}}\left(q, \Delta q_1\right), \Delta q_2\right) \ldots, \Delta q_n\right)$$

Proof

R1 Let $z = \tau_{\mathrm{q}}\left(q, \sigma_{\mathrm{q}}\left(\Delta q_1, \Delta q_2, \ldots, \Delta q_n\right)\right)$

R2 Let $y = \tau_{\mathrm{q}}\left(\ldots \tau_{\mathrm{q}}\left(\tau_{\mathrm{q}}\left(\tau_{\mathrm{q}}\left(q, \Delta q_1\right), \Delta q_2\right), \Delta q_3\right) \ldots, \Delta q_n\right)$

R3 R1 & 475 $\Rightarrow$ $z = \tau_{\mathrm{q}}\left(q, \left[\begin{array}{l} \sigma_{\mathrm{c}}\left(\Delta\,\mathrm{c}\left(\Delta q_1\right), \Delta\,\mathrm{c}\left(\Delta q_2\right), \ldots \Delta\,\mathrm{c}\left(\Delta q_n\right)\right), \\ \sigma_{\mathrm{m}}\left(\Delta\,\mathrm{m}\left(\Delta q_1\right), \Delta\,\mathrm{m}\left(\Delta q_2\right), \ldots \Delta\,\mathrm{m}\left(\Delta q_n\right)\right) \end{array}\right]\right)$

R4 R2 & 417 $\Rightarrow$ $y = \tau_{\mathrm{q}}\left(\ldots \tau_{\mathrm{q}}\left(\tau_{\mathrm{q}}\left(\left[\begin{array}{l} \tau_{\mathrm{c}}\left(\mathrm{c}\left(q\right), \Delta\,\mathrm{c}\left(\Delta q_1\right)\right), \\ \tau_{\mathrm{m}}\left(\mathrm{m}\left(q\right), \Delta\,\mathrm{m}\left(\Delta q_1\right)\right) \end{array}\right], \Delta q_2\right), \Delta q_3\right) \ldots, \Delta q_n\right)$

$$= \tau_{\mathrm{q}}\left(\ldots \tau_{\mathrm{q}}\left(\left[\begin{array}{l} \tau_{\mathrm{c}}\left(\begin{array}{l} \mathrm{c}\left(\left[\begin{array}{l} \tau_{\mathrm{c}}\left(\mathrm{c}\left(q\right), \Delta\,\mathrm{c}\left(\Delta q_1\right)\right), \\ \tau_{\mathrm{m}}\left(\mathrm{m}\left(q\right), \Delta\,\mathrm{m}\left(\Delta q_1\right)\right) \end{array}\right]\right), \\ \Delta\,\mathrm{c}\left(\Delta q_2\right) \end{array}\right), \\ \tau_{\mathrm{m}}\left(\begin{array}{l} \mathrm{m}\left(\left[\begin{array}{l} \tau_{\mathrm{c}}\left(\mathrm{c}\left(q\right), \Delta\,\mathrm{c}\left(\Delta q_1\right)\right), \\ \tau_{\mathrm{m}}\left(\mathrm{m}\left(q\right), \Delta\,\mathrm{m}\left(\Delta q_1\right)\right) \end{array}\right]\right), \\ \Delta\,\mathrm{m}\left(\Delta q_2\right) \end{array}\right) \end{array}\right], \Delta q_3\right) \ldots, \Delta q_n\right)$$

R5 R4, 106 & 108 $\Rightarrow$ $y = \tau_{\mathrm{q}}\left(\ldots \tau_{\mathrm{q}}\left(\left[\begin{array}{l} \tau_{\mathrm{c}}\left(\tau_{\mathrm{c}}\left(\mathrm{c}\left(q\right), \Delta\,\mathrm{c}\left(\Delta q_1\right)\right), \Delta\,\mathrm{c}\left(\Delta q_2\right)\right), \\ \tau_{\mathrm{m}}\left(\tau_{\mathrm{m}}\left(\mathrm{m}\left(q\right), \Delta\,\mathrm{m}\left(\Delta q_1\right)\right), \Delta\,\mathrm{m}\left(\Delta q_2\right)\right) \end{array}\right], \Delta q_3\right) \ldots, \Delta q_n\right)$

R6 R5 & 417 $\Rightarrow$

$$y = \tau_{\mathrm{q}}\left(\begin{array}{l} \ldots \left[\begin{array}{l} \tau_{\mathrm{c}}\left(\begin{array}{l} \mathrm{c}\left(\left[\begin{array}{l} \tau_{\mathrm{c}}\left(\begin{array}{l} \tau_{\mathrm{c}}\left(\mathrm{c}\left(q\right), \Delta\,\mathrm{c}\left(\Delta q_1\right)\right), \\ \Delta\,\mathrm{c}\left(\Delta q_2\right) \end{array}\right), \\ \tau_{\mathrm{m}}\left(\begin{array}{l} \tau_{\mathrm{m}}\left(\mathrm{m}\left(q\right), \Delta\,\mathrm{m}\left(\Delta q_1\right)\right), \\ \Delta\,\mathrm{m}\left(\Delta q_2\right) \end{array}\right) \end{array}\right]\right), \\ \Delta\,\mathrm{c}\left(\Delta q_3\right) \end{array}\right), \\ \tau_{\mathrm{m}}\left(\begin{array}{l} \mathrm{m}\left(\left[\begin{array}{l} \tau_{\mathrm{c}}\left(\begin{array}{l} \tau_{\mathrm{c}}\left(\mathrm{c}\left(q\right), \Delta\,\mathrm{c}\left(\Delta q_1\right)\right), \\ \Delta\,\mathrm{c}\left(\Delta q_2\right) \end{array}\right), \\ \tau_{\mathrm{m}}\left(\begin{array}{l} \tau_{\mathrm{m}}\left(\mathrm{m}\left(q\right), \Delta\,\mathrm{m}\left(\Delta q_1\right)\right), \\ \Delta\,\mathrm{m}\left(\Delta q_2\right) \end{array}\right) \end{array}\right]\right), \\ \Delta\,\mathrm{m}\left(\Delta q_3\right) \end{array}\right) \end{array}\right] \ldots, \\ \Delta q_n \end{array}\right)$$

R7 R6, 106 & 108 $\Rightarrow$

$$y = \tau_{\mathrm{q}}\left(\begin{array}{l} \ldots \left[\begin{array}{l} \tau_{\mathrm{c}}\left(\begin{array}{l} \tau_{\mathrm{c}}\left(\tau_{\mathrm{c}}\left(\mathrm{c}\left(q\right), \Delta\,\mathrm{c}\left(\Delta q_1\right)\right), \Delta\,\mathrm{c}\left(\Delta q_2\right)\right), \\ \Delta\,\mathrm{c}\left(\Delta q_3\right) \end{array}\right), \\ \tau_{\mathrm{m}}\left(\begin{array}{l} \tau_{\mathrm{m}}\left(\tau_{\mathrm{m}}\left(\mathrm{m}\left(q\right), \Delta\,\mathrm{m}\left(\Delta q_1\right)\right), \Delta\,\mathrm{m}\left(\Delta q_2\right)\right), \\ \Delta\,\mathrm{m}\left(\Delta q_3\right) \end{array}\right) \end{array}\right] \ldots, \\ \Delta q_n \end{array}\right)$$

R8 R4 to R7, 106, 108 & 417 $\Rightarrow$ $$y = \begin{bmatrix} \tau_{\mathrm{c}} \left(\begin{array}{l} \ldots \tau_{\mathrm{c}} \left(\begin{array}{l} \tau_{\mathrm{c}} \left(\begin{array}{l} \tau_{\mathrm{c}} \left(\mathrm{c} \left(q \right), \Delta \, \mathrm{c} \left(\Delta q_1 \right) \right), \\ \Delta \, \mathrm{c} \left(\Delta q_2 \right) \end{array} \right), \\ \Delta \, \mathrm{c} \left(\Delta q_3 \right) \end{array} \right) \ldots, \\ \Delta \, \mathrm{c} \left(\Delta q_n \right) \end{array} \right), \\ \tau_{\mathrm{m}} \left(\begin{array}{l} \ldots \tau_{\mathrm{m}} \left(\begin{array}{l} \tau_{\mathrm{m}} \left(\begin{array}{l} \tau_{\mathrm{m}} \left(\mathrm{m} \left(q \right), \Delta \, \mathrm{m} \left(\Delta q_1 \right) \right), \\ \Delta \, \mathrm{m} \left(\Delta q_2 \right) \end{array} \right), \\ \Delta \, \mathrm{m} \left(\Delta q_3 \right) \end{array} \right) \ldots, \\ \Delta \, \mathrm{m} \left(\Delta q_n \right) \end{array} \right) \end{bmatrix}$$

R9 R8, 448 & 462 $\Rightarrow$ $$y = \begin{bmatrix} \tau_{\mathrm{c}} \left(\mathrm{c} \left(q \right), \sigma_{\mathrm{c}} \left(\Delta \, \mathrm{c} \left(\Delta q_1 \right), \Delta \, \mathrm{c} \left(\Delta q_2 \right), \ldots, \Delta \, \mathrm{c} \left(\Delta q_n \right) \right) \right), \\ \tau_{\mathrm{m}} \left(\mathrm{m} \left(q \right), \sigma_{\mathrm{m}} \left(\Delta \, \mathrm{m} \left(\Delta q_1 \right), \Delta \, \mathrm{m} \left(\Delta q_2 \right), \ldots, \Delta \, \mathrm{m} \left(\Delta q_n \right) \right) \right) \end{bmatrix}$$

R10 R3 & 417 $\Rightarrow$ $$z = \begin{bmatrix} \tau_{\mathrm{c}} \left(\mathrm{c} \left(q \right), \Delta \, \mathrm{c} \left(\begin{bmatrix} \sigma_{\mathrm{c}} \left(\Delta \, \mathrm{c} \left(\Delta q_1 \right), \Delta \, \mathrm{c} \left(\Delta q_2 \right), \ldots \Delta \, \mathrm{c} \left(\Delta q_n \right) \right), \\ \sigma_{\mathrm{m}} \left(\Delta \, \mathrm{m} \left(\Delta q_1 \right), \Delta \, \mathrm{m} \left(\Delta q_2 \right), \ldots \Delta \, \mathrm{m} \left(\Delta q_n \right) \right) \end{bmatrix} \right) \right), \\ \tau_{\mathrm{m}} \left(\mathrm{m} \left(q \right), \Delta \, \mathrm{m} \left(\begin{bmatrix} \sigma_{\mathrm{c}} \left(\Delta \, \mathrm{c} \left(\Delta q_1 \right), \Delta \, \mathrm{c} \left(\Delta q_2 \right), \ldots \Delta \, \mathrm{c} \left(\Delta q_n \right) \right), \\ \sigma_{\mathrm{m}} \left(\Delta \, \mathrm{m} \left(\Delta q_1 \right), \Delta \, \mathrm{m} \left(\Delta q_2 \right), \ldots \Delta \, \mathrm{m} \left(\Delta q_n \right) \right) \end{bmatrix} \right) \right) \end{bmatrix}$$

R11 R10, 300 & 303 $\Rightarrow$ $$z = \begin{bmatrix} \tau_{\mathrm{c}} \left(\mathrm{c} \left(q \right), \sigma_{\mathrm{c}} \left(\Delta \, \mathrm{c} \left(\Delta q_1 \right), \Delta \, \mathrm{c} \left(\Delta q_2 \right), \ldots \Delta \, \mathrm{c} \left(\Delta q_n \right) \right) \right), \\ \tau_{\mathrm{m}} \left(\mathrm{m} \left(q \right), \sigma_{\mathrm{m}} \left(\Delta \, \mathrm{m} \left(\Delta q_1 \right), \Delta \, \mathrm{m} \left(\Delta q_2 \right), \ldots \Delta \, \mathrm{m} \left(\Delta q_n \right) \right) \right) \end{bmatrix}$$

R12 R1, R2, R9 & R11 $\Rightarrow$ $\tau_{\mathrm{q}} \left(q, \sigma_{\mathrm{q}} \left(\Delta q_1, \Delta q_2, \ldots, \Delta q_n \right) \right) = \tau_{\mathrm{q}} \left(\ldots \tau_{\mathrm{q}} \left(\tau_{\mathrm{q}} \left(q, \Delta q_1 \right), \Delta q_2 \right) \ldots, \Delta q_n \right)$

Inversion of chromamorph intervals

Definition 477 (Definition of $\iota_{\mathrm{q}} \left(\Delta q \right)$) *If ψ is a pitch system and Δq is a chromamorph interval in ψ and q is a chromamorph in ψ then $\iota_{\mathrm{q}} \left(\Delta q \right)$ is the chromamorph interval that satisfies the following equation*

$$\tau_{\mathrm{q}} \left(\tau_{\mathrm{q}} \left(q, \Delta q \right), \iota_{\mathrm{q}} \left(\Delta q \right) \right) = q$$

Definition 478 (Inversional equivalence of chromamorph intervals) *If ψ is a pitch system and Δq_1 and Δq_2 are chromamorph intervals in ψ then Δq_1 and Δq_2 are* inversionally equivalent *if and only if*

$$\left(\iota_{\mathrm{q}} \left(\Delta q_1 \right) = \Delta q_2 \right) \vee \left(\Delta q_1 = \Delta q_2 \right)$$

The fact that two chromamorph intervals are inversionally equivalent is denoted as follows:

$$\Delta q_1 \equiv_{\iota} \Delta q_2$$

Theorem 479 *If*

$$\psi = \left[\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0} \right]$$

is a pitch system and Δq is a chromamorph interval in ψ then

$$\iota_{\mathrm{q}} \left(\Delta q \right) = \left[\iota_{\mathrm{c}} \left(\Delta \, \mathrm{c} \left(\Delta q \right) \right), \iota_{\mathrm{m}} \left(\Delta \, \mathrm{m} \left(\Delta q \right) \right) \right]$$

Proof

R1	477	$\Rightarrow$	$\tau_{\mathrm{q}}\left(\tau_{\mathrm{q}}\left(q, \Delta q\right), \iota_{\mathrm{q}}\left(\Delta q\right)\right) = q$
R2	417 & R1	$\Rightarrow$	$q = \tau_{\mathrm{q}}\left(\left[\tau_{\mathrm{c}}\left(\mathrm{c}\left(q\right), \Delta\,\mathrm{c}\left(\Delta q\right)\right), \tau_{\mathrm{m}}\left(\mathrm{m}\left(q\right), \Delta\,\mathrm{m}\left(\Delta q\right)\right)\right], \iota_{\mathrm{q}}\left(\Delta q\right)\right)$ $= \left[\begin{array}{l}\tau_{\mathrm{c}}\left(\mathrm{c}\left(\left[\tau_{\mathrm{c}}\left(\mathrm{c}\left(q\right), \Delta\,\mathrm{c}\left(\Delta q\right)\right), \tau_{\mathrm{m}}\left(\mathrm{m}\left(q\right), \Delta\,\mathrm{m}\left(\Delta q\right)\right)\right]\right), \Delta\,\mathrm{c}\left(\iota_{\mathrm{q}}\left(\Delta q\right)\right)\right), \\ \tau_{\mathrm{m}}\left(\mathrm{m}\left(\left[\tau_{\mathrm{c}}\left(\mathrm{c}\left(q\right), \Delta\,\mathrm{c}\left(\Delta q\right)\right), \tau_{\mathrm{m}}\left(\mathrm{m}\left(q\right), \Delta\,\mathrm{m}\left(\Delta q\right)\right)\right]\right), \Delta\,\mathrm{m}\left(\iota_{\mathrm{q}}\left(\Delta q\right)\right)\right)\end{array}\right]$
R3	R2, 106 & 108	$\Rightarrow$	$q = \left[\begin{array}{l}\tau_{\mathrm{c}}\left(\tau_{\mathrm{c}}\left(\mathrm{c}\left(q\right), \Delta\,\mathrm{c}\left(\Delta q\right)\right), \Delta\,\mathrm{c}\left(\iota_{\mathrm{q}}\left(\Delta q\right)\right)\right), \\ \tau_{\mathrm{m}}\left(\tau_{\mathrm{m}}\left(\mathrm{m}\left(q\right), \Delta\,\mathrm{m}\left(\Delta q\right)\right), \Delta\,\mathrm{m}\left(\iota_{\mathrm{q}}\left(\Delta q\right)\right)\right)\end{array}\right]$
R4	R3 & 106	$\Rightarrow$	$\mathrm{c}\left(q\right) = \tau_{\mathrm{c}}\left(\tau_{\mathrm{c}}\left(\mathrm{c}\left(q\right), \Delta\,\mathrm{c}\left(\Delta q\right)\right), \Delta\,\mathrm{c}\left(\iota_{\mathrm{q}}\left(\Delta q\right)\right)\right)$
R5	R3 & 108	$\Rightarrow$	$\mathrm{m}\left(q\right) = \tau_{\mathrm{m}}\left(\tau_{\mathrm{m}}\left(\mathrm{m}\left(q\right), \Delta\,\mathrm{m}\left(\Delta q\right)\right), \Delta\,\mathrm{m}\left(\iota_{\mathrm{q}}\left(\Delta q\right)\right)\right)$
R6	R4 & 449	$\Rightarrow$	$\Delta\,\mathrm{c}\left(\iota_{\mathrm{q}}\left(\Delta q\right)\right) = \iota_{\mathrm{c}}\left(\Delta\,\mathrm{c}\left(\Delta q\right)\right)$
R7	R5 & 463	$\Rightarrow$	$\Delta\,\mathrm{m}\left(\iota_{\mathrm{q}}\left(\Delta q\right)\right) = \iota_{\mathrm{m}}\left(\Delta\,\mathrm{m}\left(\Delta q\right)\right)$
R8	305	$\Rightarrow$	$\iota_{\mathrm{q}}\left(\Delta q\right) = \left[\Delta\,\mathrm{c}\left(\iota_{\mathrm{q}}\left(\Delta q\right)\right), \Delta\,\mathrm{m}\left(\iota_{\mathrm{q}}\left(\Delta q\right)\right)\right]$
R9	R6, R7 & R8	$\Rightarrow$	$\iota_{\mathrm{q}}\left(\Delta q\right) = \left[\iota_{\mathrm{c}}\left(\Delta\,\mathrm{c}\left(\Delta q\right)\right), \iota_{\mathrm{m}}\left(\Delta\,\mathrm{m}\left(\Delta q\right)\right)\right]$

Theorem 480 *If ψ is a pitch system and Δq, Δq_1 and Δq_2 are chromamorph intervals in ψ then*

$$\left(\Delta q_1 = \iota_{\mathrm{q}}\left(\Delta q\right)\right) \wedge \left(\Delta q_2 = \iota_{\mathrm{q}}\left(\Delta q\right)\right) \Rightarrow \left(\Delta q_1 = \Delta q_2\right)$$

Proof

R1	Let		$\Delta q_1 = \iota_{\mathrm{q}}\left(\Delta q\right)$
R2	Let		$\Delta q_2 = \iota_{\mathrm{q}}\left(\Delta q\right)$
R3	R1 & 477	$\Rightarrow$	$\tau_{\mathrm{q}}\left(\tau_{\mathrm{q}}\left(q, \Delta q\right), \Delta q_1\right) = q$
R4	R2 & 477	$\Rightarrow$	$\tau_{\mathrm{q}}\left(\tau_{\mathrm{q}}\left(q, \Delta q\right), \Delta q_2\right) = q$
R5	R3 & R4	$\Rightarrow$	$\tau_{\mathrm{q}}\left(\tau_{\mathrm{q}}\left(q, \Delta q\right), \Delta q_1\right) = \tau_{\mathrm{q}}\left(\tau_{\mathrm{q}}\left(q, \Delta q\right), \Delta q_2\right)$
R6	R5 & 420	$\Rightarrow$	$\Delta q_1 = \Delta q_2$
R7	R1 to R6	$\Rightarrow$	$\left(\Delta q_1 = \iota_{\mathrm{q}}\left(\Delta q\right)\right) \wedge \left(\Delta q_2 = \iota_{\mathrm{q}}\left(\Delta q\right)\right) \Rightarrow \left(\Delta q_1 = \Delta q_2\right)$

Exponentiation of chromamorph intervals

Definition 481 (Definition of $\epsilon_{\mathrm{q},n}\left(\Delta q\right)$) *Given that:*

1. *ψ is a pitch system;*
2. *q is a chromamorph in ψ;*
3. *Δq is a chromamorph interval in ψ;*
4. *n is an integer;*
5. *k is an integer and $1 \leq k \leq \mathrm{abs}(n)$;*
6. *$\Delta q_{1,k} = \Delta q$ for all k; and*
7. *$\Delta q_{2,k} = \iota_{\mathrm{q}}(\Delta q)$ for all k;*

then $\epsilon_{\mathrm{q},n}(\Delta q)$ returns a chromamorph interval that satisfies the following equation:

$$\tau_{\mathrm{q}}(q, \epsilon_{\mathrm{q},n}(\Delta q)) = \begin{cases} \tau_{\mathrm{q}}(q, \sigma_{\mathrm{q}}(\Delta q_{1,1}, \Delta q_{1,2}, \ldots \Delta q_{1,n})) & \text{if} \quad n > 0 \\ q & \text{if} \quad n = 0 \\ \tau_{\mathrm{q}}(q, \sigma_{\mathrm{q}}(\Delta q_{2,1}, \Delta q_{2,2}, \ldots \Delta q_{2,-n})) & \text{if} \quad n < 0 \end{cases}$$

Theorem 482 (Formula for $\epsilon_{\mathrm{q},n}(\Delta q)$) *If*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

is a pitch system and Δq is a chromamorph interval in ψ and n is an integer then

$$\epsilon_{\mathrm{q},n}(\Delta q) = [\epsilon_{\mathrm{c},n}(\Delta\,\mathrm{c}(\Delta q)), \epsilon_{\mathrm{m},n}(\Delta\,\mathrm{m}(\Delta q))]$$

Proof

R1	Let		$n \in \mathbb{Z}$
R2	R1 & 454	$\Rightarrow$	$\epsilon_{\mathrm{c},n}\left(\Delta\,\mathrm{c}\left(\Delta q\right)\right) = \left(n \times \Delta\,\mathrm{c}\left(\Delta q\right)\right) \bmod \mu_{\mathrm{c}}$
R3	R1 & 468	$\Rightarrow$	$\epsilon_{\mathrm{m},n}\left(\Delta\,\mathrm{m}\left(\Delta q\right)\right) = \left(n \times \Delta\,\mathrm{m}\left(\Delta q\right)\right) \bmod \mu_{\mathrm{m}}$
R4	Let		$\left(1 \leq k \leq \mathrm{abs}\left(n\right)\right) \wedge \left(k \in \mathbb{Z}\right)$
R5	Let		$\Delta q_{1,k} = \Delta q$ for all k
R6	Let		$\Delta q_{2,k} = \iota_{\mathrm{q}}\left(\Delta q\right)$ for all k
R7	R1, R4, R5, R6 & 481	$\Rightarrow$	$\tau_{\mathrm{q}}\left(q, \epsilon_{\mathrm{q},n}\left(\Delta q\right)\right) = \begin{cases} \tau_{\mathrm{q}}\left(q, \sigma_{\mathrm{q}}\left(\Delta q_{1,1}, \Delta q_{1,2}, \ldots \Delta q_{1,n}\right)\right) & \text{if} \quad n > 0 \\ q & \text{if} \quad n = 0 \\ \tau_{\mathrm{q}}\left(q, \sigma_{\mathrm{q}}\left(\Delta q_{2,1}, \Delta q_{2,2}, \ldots \Delta q_{2,-n}\right)\right) & \text{if} \quad n < 0 \end{cases}$
R8	475	$\Rightarrow$	$\sigma_{\mathrm{q}}\left(\Delta q_{1,1}, \Delta q_{1,2}, \ldots \Delta q_{1,n}\right)$
			$= \left[\begin{array}{l} \sigma_{\mathrm{c}}\left(\Delta\,\mathrm{c}\left(\Delta q_{1,1}\right), \Delta\,\mathrm{c}\left(\Delta q_{1,2}\right), \ldots \Delta\,\mathrm{c}\left(\Delta q_{1,n}\right)\right), \\ \sigma_{\mathrm{m}}\left(\Delta\,\mathrm{m}\left(\Delta q_{1,1}\right), \Delta\,\mathrm{m}\left(\Delta q_{1,2}\right), \ldots \Delta\,\mathrm{m}\left(\Delta q_{1,n}\right)\right) \end{array}\right]$ where $n > 0$
R9	447, 461 & R8	$\Rightarrow$	$\sigma_{\mathrm{q}}\left(\Delta q_{1,1}, \Delta q_{1,2}, \ldots \Delta q_{1,n}\right)$
			$= \left[\left(\sum_{k=1}^{n} \Delta\,\mathrm{c}\left(\Delta q_{1,k}\right)\right) \bmod \mu_{\mathrm{c}}, \left(\sum_{k=1}^{n} \Delta\,\mathrm{m}\left(\Delta q_{1,k}\right)\right) \bmod \mu_{\mathrm{m}}\right]$ where $n > 0$
R10	R9 & R5	$\Rightarrow$	$\sigma_{\mathrm{q}}\left(\Delta q_{1,1}, \Delta q_{1,2}, \ldots \Delta q_{1,n}\right)$
			$= \left[\left(n \times \Delta\,\mathrm{c}\left(\Delta q\right)\right) \bmod \mu_{\mathrm{c}}, \left(n \times \Delta\,\mathrm{m}\left(\Delta q\right)\right) \bmod \mu_{\mathrm{m}}\right]$ where $n > 0$
R11	R10, 454 & 468	$\Rightarrow$	$\sigma_{\mathrm{q}}\left(\Delta q_{1,1}, \Delta q_{1,2}, \ldots \Delta q_{1,n}\right)$
			$= \left[\epsilon_{\mathrm{c},n}\left(\Delta\,\mathrm{c}\left(\Delta q\right)\right), \epsilon_{\mathrm{m},n}\left(\Delta\,\mathrm{m}\left(\Delta q\right)\right)\right]$ where $n > 0$
R12	R7 & R11	$\Rightarrow$	$\tau_{\mathrm{q}}\left(q, \epsilon_{\mathrm{q},n}\left(\Delta q\right)\right) = \tau_{\mathrm{q}}\left(q, \epsilon_{\mathrm{c},n}\left(\Delta\,\mathrm{c}\left(\Delta q\right)\right), \epsilon_{\mathrm{m},n}\left(\Delta\,\mathrm{m}\left(\Delta q\right)\right)\right)$ where $n > 0$
R13	454 & 468	$\Rightarrow$	$\tau_{\mathrm{q}}\left(q, \epsilon_{\mathrm{c},0}\left(\Delta\,\mathrm{c}\left(\Delta q\right)\right), \epsilon_{\mathrm{m},0}\left(\Delta\,\mathrm{m}\left(\Delta q\right)\right)\right)$
			$= \tau_{\mathrm{q}}\left(q, \left[\left(0 \times \Delta\,\mathrm{c}\left(\Delta q\right)\right) \bmod \mu_{\mathrm{c}}, \left(0 \times \Delta\,\mathrm{m}\left(\Delta q\right)\right) \bmod \mu_{\mathrm{m}}\right]\right)$
			$= \tau_{\mathrm{q}}\left(q, \left[0, 0\right]\right)$

R14 R13, 300, 303 & 417 $\Rightarrow$ $\tau_{\mathrm{q}}\left(q,\left[\epsilon_{\mathrm{c},0}\left(\Delta\,\mathrm{c}\left(\Delta q\right)\right),\epsilon_{\mathrm{m},0}\left(\Delta\,\mathrm{m}\left(\Delta q\right)\right)\right]\right)$

$$=\left[\tau_{\mathrm{c}}\left(\mathrm{c}\left(q\right),0\right),\tau_{\mathrm{m}}\left(\mathrm{m}\left(q\right),0\right)\right]$$

R15 R14, 407 & 412 $\Rightarrow$ $\tau_{\mathrm{q}}\left(q,\left[\epsilon_{\mathrm{c},0}\left(\Delta\,\mathrm{c}\left(\Delta q\right)\right),\epsilon_{\mathrm{m},0}\left(\Delta\,\mathrm{m}\left(\Delta q\right)\right)\right]\right)=\left[\mathrm{c}\left(q\right)\bmod\mu_{\mathrm{c}},\mathrm{m}\left(q\right)\bmod\mu_{\mathrm{m}}\right]$

R16 R15, 73 & 78 $\Rightarrow$ $\tau_{\mathrm{q}}\left(q,\left[\epsilon_{\mathrm{c},0}\left(\Delta\,\mathrm{c}\left(\Delta q\right)\right),\epsilon_{\mathrm{m},0}\left(\Delta\,\mathrm{m}\left(\Delta q\right)\right)\right]\right)=\left[\mathrm{c}\left(q\right),\mathrm{m}\left(q\right)\right]$

R17 R16 & 109 $\Rightarrow$ $\tau_{\mathrm{q}}\left(q,\left[\epsilon_{\mathrm{c},0}\left(\Delta\,\mathrm{c}\left(\Delta q\right)\right),\epsilon_{\mathrm{m},0}\left(\Delta\,\mathrm{m}\left(\Delta q\right)\right)\right]\right)=q$

R18 R7 & R17 $\Rightarrow$ $\tau_{\mathrm{q}}\left(q,\epsilon_{\mathrm{q},n}\left(\Delta q\right)\right)=\tau_{\mathrm{q}}\left(q,\left[\epsilon_{\mathrm{c},n}\left(\Delta\,\mathrm{c}\left(\Delta q\right)\right),\epsilon_{\mathrm{m},n}\left(\Delta\,\mathrm{m}\left(\Delta q\right)\right)\right]\right)$ where $n=0$

R19 475 $\Rightarrow$ $\sigma_{\mathrm{q}}\left(\Delta q_{2,1},\Delta q_{2,2},\ldots\Delta q_{2,-n}\right)$

$$=\left[\begin{array}{l}\sigma_{\mathrm{c}}\left(\Delta\,\mathrm{c}\left(\Delta q_{2,1}\right),\Delta\,\mathrm{c}\left(\Delta q_{2,2}\right),\ldots\Delta\,\mathrm{c}\left(\Delta q_{2,-n}\right)\right),\\\sigma_{\mathrm{m}}\left(\Delta\,\mathrm{m}\left(\Delta q_{2,1}\right),\Delta\,\mathrm{m}\left(\Delta q_{2,2}\right),\ldots\Delta\,\mathrm{m}\left(\Delta q_{2,-n}\right)\right)\end{array}\right]\text{ where }n<0$$

R20 R19, 447 & 461 $\Rightarrow$ $\sigma_{\mathrm{q}}\left(\Delta q_{2,1},\Delta q_{2,2},\ldots\Delta q_{2,-n}\right)$

$$=\left[\begin{array}{l}\left(\sum_{k=1}^{-n}\Delta\,\mathrm{c}\left(\Delta q_{2,k}\right)\right)\bmod\mu_{\mathrm{c}},\\\left(\sum_{k=1}^{-n}\Delta\,\mathrm{m}\left(\Delta q_{2,k}\right)\right)\bmod\mu_{\mathrm{m}}\end{array}\right]\text{ where }n<0$$

R21 R6 & R20 $\Rightarrow$ $\sigma_{\mathrm{q}}\left(\Delta q_{2,1},\Delta q_{2,2},\ldots\Delta q_{2,-n}\right)$

$$=\left[\begin{array}{l}\left(-n\times\Delta\,\mathrm{c}\left(\iota_{\mathrm{q}}\left(\Delta q\right)\right)\right)\bmod\mu_{\mathrm{c}},\\\left(-n\times\Delta\,\mathrm{m}\left(\iota_{\mathrm{q}}\left(\Delta q\right)\right)\right)\bmod\mu_{\mathrm{m}}\end{array}\right]\text{ where }n<0$$

R22 R21, 479, 300 & 303 $\Rightarrow$ $\sigma_{\mathrm{q}}\left(\Delta q_{2,1},\Delta q_{2,2},\ldots\Delta q_{2,-n}\right)$

$$=\left[\begin{array}{l}\left(-n\times\iota_{\mathrm{c}}\left(\Delta\,\mathrm{c}\left(\Delta q\right)\right)\right)\bmod\mu_{\mathrm{c}},\\\left(-n\times\iota_{\mathrm{m}}\left(\Delta\,\mathrm{m}\left(\Delta q\right)\right)\right)\bmod\mu_{\mathrm{m}}\end{array}\right]\text{ where }n<0$$

R23 R22, 455 & 469 $\Rightarrow$ $\sigma_{\mathrm{q}}\left(\Delta q_{2,1},\Delta q_{2,2},\ldots\Delta q_{2,-n}\right)$

$$=\left[\begin{array}{l}\left(-n\times\epsilon_{\mathrm{c},-1}\left(\Delta\,\mathrm{c}\left(\Delta q\right)\right)\right)\bmod\mu_{\mathrm{c}},\\\left(-n\times\epsilon_{\mathrm{m},-1}\left(\Delta\,\mathrm{m}\left(\Delta q\right)\right)\right)\bmod\mu_{\mathrm{m}}\end{array}\right]\text{ where }n<0$$

R24 R23, 454 & 468 $\Rightarrow$ $\sigma_{\mathrm{q}}\left(\Delta q_{2,1},\Delta q_{2,2},\ldots\Delta q_{2,-n}\right)$

$$=\left[\begin{array}{l}\left(-n\times\left(-\Delta\,\mathrm{c}\left(\Delta q\right)\bmod\mu_{\mathrm{c}}\right)\right)\bmod\mu_{\mathrm{c}},\\\left(-n\times\left(-\Delta\,\mathrm{m}\left(\Delta q\right)\bmod\mu_{\mathrm{m}}\right)\right)\bmod\mu_{\mathrm{m}}\end{array}\right]\text{ where }n<0$$

R25 R24 & 45 $\Rightarrow$ $\sigma_{\mathrm{q}}\left(\Delta q_{2,1},\Delta q_{2,2},\ldots\Delta q_{2,-n}\right)$

$$=\left[\left(-n\times\left(-\Delta\,\mathrm{c}\left(\Delta q\right)\right)\right)\bmod\mu_{\mathrm{c}},\left(-n\times\left(-\Delta\,\mathrm{m}\left(\Delta q\right)\right)\right)\bmod\mu_{\mathrm{m}}\right]$$

$$=\left[\left(n\times\Delta\,\mathrm{c}\left(\Delta q\right)\right)\bmod\mu_{\mathrm{c}},\left(n\times\Delta\,\mathrm{m}\left(\Delta q\right)\right)\bmod\mu_{\mathrm{m}}\right]\text{ where }n<0$$

R26	R25, 454 & 468	$\Rightarrow$	$\sigma_{\mathrm{q}}\left(\Delta q_{2,1}, \Delta q_{2,2}, \ldots \Delta q_{2,-n}\right)$
			$= \left[\epsilon_{\mathrm{c},n}\left(\Delta\,\mathrm{c}\left(\Delta q\right)\right), \epsilon_{\mathrm{m},n}\left(\Delta\,\mathrm{m}\left(\Delta q\right)\right)\right]$ where $n < 0$
R27	R26 & R7	$\Rightarrow$	$\tau_{\mathrm{q}}\left(q, \epsilon_{\mathrm{q},n}\left(\Delta q\right)\right) = \tau_{\mathrm{q}}\left(q, \left[\epsilon_{\mathrm{c},n}\left(\Delta\,\mathrm{c}\left(\Delta q\right)\right), \epsilon_{\mathrm{m},n}\left(\Delta\,\mathrm{m}\left(\Delta q\right)\right)\right]\right)$ where $n < 0$
R28	R12, R18 & R27	$\Rightarrow$	$\tau_{\mathrm{q}}\left(q, \epsilon_{\mathrm{q},n}\left(\Delta q\right)\right) = \tau_{\mathrm{q}}\left(q, \left[\epsilon_{\mathrm{c},n}\left(\Delta\,\mathrm{c}\left(\Delta q\right)\right), \epsilon_{\mathrm{m},n}\left(\Delta\,\mathrm{m}\left(\Delta q\right)\right)\right]\right)$ for all $n \in \mathbb{Z}$
R29	R28 & 420	$\Rightarrow$	$\epsilon_{\mathrm{q},n}\left(\Delta q\right) = \left[\epsilon_{\mathrm{c},n}\left(\Delta\,\mathrm{c}\left(\Delta q\right)\right), \epsilon_{\mathrm{m},n}\left(\Delta\,\mathrm{m}\left(\Delta q\right)\right)\right]$

Theorem 483 *If*

$$\psi = \left[\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}\right]$$

is a pitch system and Δq is any chromamorph interval in ψ then

$$\iota_{\mathrm{q}}\left(\Delta q\right) = \epsilon_{\mathrm{q},-1}\left(\Delta q\right)$$

Proof

R1	479	$\Rightarrow$	$\iota_{\mathrm{q}}\left(\Delta q\right) = \left[\iota_{\mathrm{c}}\left(\Delta\,\mathrm{c}\left(\Delta q\right)\right), \iota_{\mathrm{m}}\left(\Delta\,\mathrm{m}\left(\Delta q\right)\right)\right]$
R2	482	$\Rightarrow$	$\epsilon_{\mathrm{q},-1}\left(\Delta q\right) = \left[\epsilon_{\mathrm{c},-1}\left(\Delta\,\mathrm{c}\left(\Delta q\right)\right), \epsilon_{\mathrm{m},-1}\left(\Delta\,\mathrm{m}\left(\Delta q\right)\right)\right]$
R3	R1, 451 & 465	$\Rightarrow$	$\iota_{\mathrm{q}}\left(\Delta q\right) = \left[\left(-\Delta\,\mathrm{c}\left(\Delta q\right)\right) \bmod \mu_{\mathrm{c}}, \left(-\Delta\,\mathrm{m}\left(\Delta q\right)\right) \bmod \mu_{\mathrm{m}}\right]$
R4	R2, 454 & 468	$\Rightarrow$	$\epsilon_{\mathrm{q},-1}\left(\Delta q\right) = \left[\left(-\Delta\,\mathrm{c}\left(\Delta q\right)\right) \bmod \mu_{\mathrm{c}}, \left(-\Delta\,\mathrm{m}\left(\Delta q\right)\right) \bmod \mu_{\mathrm{m}}\right]$
R5	R3 & R4	$\Rightarrow$	$\iota_{\mathrm{q}}\left(\Delta q\right) = \epsilon_{\mathrm{q},-1}\left(\Delta q\right)$

Theorem 484 *If*

$$\psi = \left[\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}\right]$$

is a pitch system, $n_1, n_2, \ldots n_k$ is a collection of integers and Δq is a chromamorph interval in ψ then

$$\epsilon_{\mathrm{q},n_k}\left(\ldots \epsilon_{\mathrm{q},n_2}\left(\epsilon_{\mathrm{q},n_1}\left(\Delta q\right)\right)\ldots\right) = \epsilon_{\mathrm{q},\prod_{j=1}^{k} n_j}\left(\Delta q\right)$$

Proof

R1			$\prod_{j=1}^{1} n_j = n_1$
R2	R1	$\Rightarrow$	$\epsilon_{\mathrm{q},n_1}\left(\Delta q\right) = \epsilon_{\mathrm{q},\prod_{j=1}^{1} n_j}\left(\Delta q\right)$
R3	R2	$\Rightarrow$	$\epsilon_{\mathrm{q},n_k}\left(\ldots \epsilon_{\mathrm{q},n_2}\left(\epsilon_{\mathrm{q},n_1}\left(\Delta q\right)\right)\ldots\right) = \epsilon_{\mathrm{q},\prod_{j=1}^{k} n_j}\left(\Delta q\right)$ when $k = 1$

R4 481 $\Rightarrow \left(\begin{array}{l} \epsilon_{\mathrm{q},n_k}\left(\ldots\epsilon_{\mathrm{q},n_2}\left(\epsilon_{\mathrm{q},n_1}\left(\Delta q\right)\right)\ldots\right)=\epsilon_{\mathrm{q},\prod_{j=1}^{k}n_j}\left(\Delta q\right) \\ \Rightarrow \epsilon_{\mathrm{q},n_{k+1}}\left(\epsilon_{\mathrm{q},n_k}\left(\ldots\epsilon_{\mathrm{q},n_2}\left(\epsilon_{\mathrm{q},n_1}\left(\Delta q\right)\right)\ldots\right)\right)=\epsilon_{\mathrm{q},n_{k+1}}\left(\epsilon_{\mathrm{q},\prod_{j=1}^{k}n_j}\left(\Delta q\right)\right) \end{array}\right)$

R5 R4 & 482 $\Rightarrow \left(\begin{array}{l} \epsilon_{\mathrm{q},n_k}\left(\ldots\epsilon_{\mathrm{q},n_2}\left(\epsilon_{\mathrm{q},n_1}\left(\Delta q\right)\right)\ldots\right)=\epsilon_{\mathrm{q},\prod_{j=1}^{k}n_j}\left(\Delta q\right) \\ \Rightarrow \epsilon_{\mathrm{q},n_{k+1}}\left(\epsilon_{\mathrm{q},n_k}\left(\ldots\epsilon_{\mathrm{q},n_2}\left(\epsilon_{\mathrm{q},n_1}\left(\Delta q\right)\right)\ldots\right)\right)=\epsilon_{\mathrm{q},n_{k+1}}\left(\left[\begin{array}{l} \epsilon_{\mathrm{c},\prod_{j=1}^{k}n_j}\left(\Delta\,\mathrm{c}\left(\Delta q\right)\right), \\ \epsilon_{\mathrm{m},\prod_{j=1}^{k}n_j}\left(\Delta\,\mathrm{m}\left(\Delta q\right)\right) \end{array}\right]\right) \end{array}\right)$

R6 R5 & 454 $\Rightarrow \left(\begin{array}{l} \epsilon_{\mathrm{q},n_k}\left(\ldots\epsilon_{\mathrm{q},n_2}\left(\epsilon_{\mathrm{q},n_1}\left(\Delta q\right)\right)\ldots\right)=\epsilon_{\mathrm{q},\prod_{j=1}^{k}n_j}\left(\Delta q\right) \\ \Rightarrow \epsilon_{\mathrm{q},n_{k+1}}\left(\epsilon_{\mathrm{q},n_k}\left(\ldots\epsilon_{\mathrm{q},n_2}\left(\epsilon_{\mathrm{q},n_1}\left(\Delta q\right)\right)\ldots\right)\right) \\ =\epsilon_{\mathrm{q},n_{k+1}}\left(\left[\begin{array}{l} \left(\prod_{j=1}^{k}n_j\times\Delta\,\mathrm{c}\left(\Delta q\right)\right)\bmod\mu_{\mathrm{c}}, \\ \left(\prod_{j=1}^{k}n_j\times\Delta\,\mathrm{m}\left(\Delta q\right)\right)\bmod\mu_{\mathrm{m}} \end{array}\right]\right) \end{array}\right)$

R7 R6, 482, 300 & 303 $\Rightarrow \left(\begin{array}{l} \epsilon_{\mathrm{q},n_k}\left(\ldots\epsilon_{\mathrm{q},n_2}\left(\epsilon_{\mathrm{q},n_1}\left(\Delta q\right)\right)\ldots\right)=\epsilon_{\mathrm{q},\prod_{j=1}^{k}n_j}\left(\Delta q\right) \\ \Rightarrow \epsilon_{\mathrm{q},n_{k+1}}\left(\epsilon_{\mathrm{q},n_k}\left(\ldots\epsilon_{\mathrm{q},n_2}\left(\epsilon_{\mathrm{q},n_1}\left(\Delta q\right)\right)\ldots\right)\right) \\ =\left[\begin{array}{l} \epsilon_{\mathrm{c},n_{k+1}}\left(\left(\prod_{j=1}^{k}n_j\times\Delta\,\mathrm{c}\left(\Delta q\right)\right)\bmod\mu_{\mathrm{c}}\right), \\ \epsilon_{\mathrm{m},n_{k+1}}\left(\left(\prod_{j=1}^{k}n_j\times\Delta\,\mathrm{m}\left(\Delta q\right)\right)\bmod\mu_{\mathrm{m}}\right) \end{array}\right] \end{array}\right)$

R8 R7, 454 & 468 $\Rightarrow \left(\begin{array}{l} \epsilon_{\mathrm{q},n_k}\left(\ldots\epsilon_{\mathrm{q},n_2}\left(\epsilon_{\mathrm{q},n_1}\left(\Delta q\right)\right)\ldots\right)=\epsilon_{\mathrm{q},\prod_{j=1}^{k}n_j}\left(\Delta q\right) \\ \Rightarrow \epsilon_{\mathrm{q},n_{k+1}}\left(\epsilon_{\mathrm{q},n_k}\left(\ldots\epsilon_{\mathrm{q},n_2}\left(\epsilon_{\mathrm{q},n_1}\left(\Delta q\right)\right)\ldots\right)\right)= \\ \left[\begin{array}{l} \left(n_{k+1}\times\left(\left(\prod_{j=1}^{k}n_j\times\Delta\,\mathrm{c}\left(\Delta q\right)\right)\bmod\mu_{\mathrm{c}}\right)\right)\bmod\mu_{\mathrm{c}}, \\ \left(n_{k+1}\times\left(\left(\prod_{j=1}^{k}n_j\times\Delta\,\mathrm{m}\left(\Delta q\right)\right)\bmod\mu_{\mathrm{m}}\right)\right)\bmod\mu_{\mathrm{m}} \end{array}\right] \end{array}\right)$

R9 R8 & 45 $\Rightarrow \left(\begin{array}{l} \epsilon_{\mathrm{q},n_k}\left(\ldots\epsilon_{\mathrm{q},n_2}\left(\epsilon_{\mathrm{q},n_1}\left(\Delta q\right)\right)\ldots\right)=\epsilon_{\mathrm{q},\prod_{j=1}^{k}n_j}\left(\Delta q\right) \\ \Rightarrow \epsilon_{\mathrm{q},n_{k+1}}\left(\epsilon_{\mathrm{q},n_k}\left(\ldots\epsilon_{\mathrm{q},n_2}\left(\epsilon_{\mathrm{q},n_1}\left(\Delta q\right)\right)\ldots\right)\right) \\ =\left[\begin{array}{l} \left(n_{k+1}\times\prod_{j=1}^{k}n_j\times\Delta\,\mathrm{c}\left(\Delta q\right)\right)\bmod\mu_{\mathrm{c}}, \\ \left(n_{k+1}\times\prod_{j=1}^{k}n_j\times\Delta\,\mathrm{m}\left(\Delta q\right)\right)\bmod\mu_{\mathrm{m}} \end{array}\right] \\ =\left[\begin{array}{l} \left(\prod_{j=1}^{k+1}n_j\times\Delta\,\mathrm{c}\left(\Delta q\right)\right)\bmod\mu_{\mathrm{c}}, \\ \left(\prod_{j=1}^{k+1}n_j\times\Delta\,\mathrm{m}\left(\Delta q\right)\right)\bmod\mu_{\mathrm{m}} \end{array}\right] \end{array}\right)$

R10 R9, 454 & 468 $\Rightarrow \left(\begin{array}{l} \epsilon_{\mathrm{q},n_k}\left(\ldots\epsilon_{\mathrm{q},n_2}\left(\epsilon_{\mathrm{q},n_1}\left(\Delta q\right)\right)\ldots\right)=\epsilon_{\mathrm{q},\prod_{j=1}^{k}n_j}\left(\Delta q\right) \\ \Rightarrow \epsilon_{\mathrm{q},n_{k+1}}\left(\epsilon_{\mathrm{q},n_k}\left(\ldots\epsilon_{\mathrm{q},n_2}\left(\epsilon_{\mathrm{q},n_1}\left(\Delta q\right)\right)\ldots\right)\right) \\ =\left[\epsilon_{\mathrm{c},\prod_{j=1}^{k+1}n_j}\left(\Delta\,\mathrm{c}\left(\Delta q\right)\right),\epsilon_{\mathrm{m},\prod_{j=1}^{k+1}n_j}\left(\Delta\,\mathrm{m}\left(\Delta q\right)\right)\right] \end{array}\right)$

R11 R10 & 482 $\Rightarrow \left(\begin{array}{l} \epsilon_{\mathrm{q},n_k}\left(\ldots\epsilon_{\mathrm{q},n_2}\left(\epsilon_{\mathrm{q},n_1}\left(\Delta q\right)\right)\ldots\right)=\epsilon_{\mathrm{q},\prod_{j=1}^{k}n_j}\left(\Delta q\right) \\ \Rightarrow \epsilon_{\mathrm{q},n_{k+1}}\left(\epsilon_{\mathrm{q},n_k}\left(\ldots\epsilon_{\mathrm{q},n_2}\left(\epsilon_{\mathrm{q},n_1}\left(\Delta q\right)\right)\ldots\right)\right)=\epsilon_{\mathrm{q},\prod_{j=1}^{k+1}n_j}\left(\Delta q\right) \end{array}\right)$

R12 R3 & R11 $\Rightarrow \epsilon_{\mathrm{q},n_k}\left(\ldots\epsilon_{\mathrm{q},n_2}\left(\epsilon_{\mathrm{q},n_1}\left(\Delta q\right)\right)\ldots\right)=\epsilon_{\mathrm{q},\prod_{j=1}^{k}n_j}\left(\Delta q\right)$ for all $k\in\mathbb{Z},k>0$.

Theorem 485 *If*

$$\psi=\left[\mu_{\mathrm{c}},\mu_{\mathrm{m}},f_0,p_{\mathrm{c},0}\right]$$

is a pitch system, n is an integer and Δq is a chromamorph interval in ψ then

$$\iota_{\mathrm{q}}\left(\epsilon_{\mathrm{q},n}\left(\Delta q\right)\right)=\epsilon_{\mathrm{q},-n}\left(\Delta q\right)$$

Proof

R1	483	$\Rightarrow$	$\iota_{\mathrm{q}}\left(\Delta q\right)=\epsilon_{\mathrm{q},-1}\left(\Delta q\right)$
R2	R1	$\Rightarrow$	$\iota_{\mathrm{q}}\left(\epsilon_{\mathrm{q},n}\left(\Delta q\right)\right)=\epsilon_{\mathrm{q},-1}\left(\epsilon_{\mathrm{q},n}\left(\Delta q\right)\right)$
R3	R2 & 484	$\Rightarrow$	$\iota_{\mathrm{q}}\left(\epsilon_{\mathrm{q},n}\left(\Delta q\right)\right)=\epsilon_{\mathrm{q},(-1\times n)}\left(\Delta q\right)=\epsilon_{\mathrm{q},-n}\left(\Delta q\right)$

Theorem 486 *If*

$$\psi=\left[\mu_{\mathrm{c}},\mu_{\mathrm{m}},f_0,p_{\mathrm{c},0}\right]$$

is a pitch system, n is an integer and Δq is a chromamorph interval in ψ then:

$$\Delta\,\mathrm{c}\left(\epsilon_{\mathrm{q},n}\left(\Delta q\right)\right)=\epsilon_{\mathrm{c},n}\left(\Delta\,\mathrm{c}\left(\Delta q\right)\right)$$

Proof

R1	482	$\Rightarrow$	$\epsilon_{\mathrm{q},n}\left(\Delta q\right)=\left[\epsilon_{\mathrm{c},n}\left(\Delta\,\mathrm{c}\left(\Delta q\right)\right),\epsilon_{\mathrm{m},n}\left(\Delta\,\mathrm{m}\left(\Delta q\right)\right)\right]$
R2	R1 & 300	$\Rightarrow$	$\Delta\,\mathrm{c}\left(\epsilon_{\mathrm{q},n}\left(\Delta q\right)\right)=\epsilon_{\mathrm{c},n}\left(\Delta\,\mathrm{c}\left(\Delta q\right)\right)$

Theorem 487 *If*

$$\psi=\left[\mu_{\mathrm{c}},\mu_{\mathrm{m}},f_0,p_{\mathrm{c},0}\right]$$

is a pitch system, n is an integer and Δq is a chromamorph interval in ψ then:

$$\Delta\,\mathrm{m}\left(\epsilon_{\mathrm{q},n}\left(\Delta q\right)\right)=\epsilon_{\mathrm{m},n}\left(\Delta\,\mathrm{m}\left(\Delta q\right)\right)$$

Proof

R1	482	$\Rightarrow$	$\epsilon_{\mathrm{q},n}\left(\Delta q\right)=\left[\epsilon_{\mathrm{c},n}\left(\Delta\,\mathrm{c}\left(\Delta q\right)\right),\epsilon_{\mathrm{m},n}\left(\Delta\,\mathrm{m}\left(\Delta q\right)\right)\right]$
R2	R1 & 303	$\Rightarrow$	$\Delta\,\mathrm{m}\left(\epsilon_{\mathrm{q},n}\left(\Delta q\right)\right)=\epsilon_{\mathrm{m},n}\left(\Delta\,\mathrm{m}\left(\Delta q\right)\right)$

Theorem 488 *If*

$$\psi=\left[\mu_{\mathrm{c}},\mu_{\mathrm{m}},f_0,p_{\mathrm{c},0}\right]$$

is a pitch system, $n_1,n_2,\ldots n_k$ is a collection of integers and Δq is a chromamorph interval in ψ then

$$\sigma_{\mathrm{q}}\left(\epsilon_{\mathrm{q},n_1}\left(\Delta q\right),\epsilon_{\mathrm{q},n_2}\left(\Delta q\right),\ldots,\epsilon_{\mathrm{q},n_k}\left(\Delta q\right)\right)=\epsilon_{\mathrm{q},\sum_{j=1}^{k}n_j}\left(\Delta q\right)$$

Proof

R1	Let		$y = \sigma_{\mathrm{q}}\left(\epsilon_{\mathrm{q},n_1}\left(\Delta q\right), \epsilon_{\mathrm{q},n_2}\left(\Delta q\right), \ldots, \epsilon_{\mathrm{q},n_k}\left(\Delta q\right)\right)$
R2	R1 & 475	$\Rightarrow$	$y = \left[\begin{array}{l}\sigma_{\mathrm{c}}\left(\Delta\,\mathrm{c}\left(\epsilon_{\mathrm{q},n_1}\left(\Delta q\right)\right), \Delta\,\mathrm{c}\left(\epsilon_{\mathrm{q},n_2}\left(\Delta q\right)\right), \ldots, \Delta\,\mathrm{c}\left(\epsilon_{\mathrm{q},n_k}\left(\Delta q\right)\right)\right), \\ \sigma_{\mathrm{m}}\left(\Delta\,\mathrm{m}\left(\epsilon_{\mathrm{q},n_1}\left(\Delta q\right)\right), \Delta\,\mathrm{m}\left(\epsilon_{\mathrm{q},n_2}\left(\Delta q\right)\right), \ldots, \Delta\,\mathrm{m}\left(\epsilon_{\mathrm{q},n_k}\left(\Delta q\right)\right)\right)\end{array}\right]$
R3	R2, 486 & 487	$\Rightarrow$	$y = \left[\begin{array}{l}\sigma_{\mathrm{c}}\left(\epsilon_{\mathrm{c},n_1}\left(\Delta\,\mathrm{c}\left(\Delta q\right)\right), \epsilon_{\mathrm{c},n_2}\left(\Delta\,\mathrm{c}\left(\Delta q\right)\right), \ldots, \epsilon_{\mathrm{c},n_k}\left(\Delta\,\mathrm{c}\left(\Delta q\right)\right)\right), \\ \sigma_{\mathrm{m}}\left(\epsilon_{\mathrm{m},n_1}\left(\Delta\,\mathrm{m}\left(\Delta q\right)\right), \epsilon_{\mathrm{m},n_2}\left(\Delta\,\mathrm{m}\left(\Delta q\right)\right), \ldots, \epsilon_{\mathrm{m},n_k}\left(\Delta\,\mathrm{m}\left(\Delta q\right)\right)\right)\end{array}\right]$
R4	R3, 458 & 472	$\Rightarrow$	$y = \left[\epsilon_{\mathrm{c},\sum_{j=1}^{k} n_j}\left(\Delta\,\mathrm{c}\left(\Delta q\right)\right), \epsilon_{\mathrm{m},\sum_{j=1}^{k} n_j}\left(\Delta\,\mathrm{m}\left(\Delta q\right)\right)\right]$
R5	R1, R4 & 482	$\Rightarrow$	$\sigma_{\mathrm{q}}\left(\epsilon_{\mathrm{q},n_1}\left(\Delta q\right), \epsilon_{\mathrm{q},n_2}\left(\Delta q\right), \ldots, \epsilon_{\mathrm{q},n_k}\left(\Delta q\right)\right) = \epsilon_{\mathrm{q},\sum_{j=1}^{k} n_j}\left(\Delta q\right)$

Exponentiation of the chromamorph tranposition function

Definition 489 (Definition of $\tau_{\mathrm{q},n}\left(q, \Delta q\right)$) *If ψ is a pitch system and q is a chromamorph in ψ and Δq is a chromamorph interval in ψ then*

$$\tau_{\mathrm{q},n}\left(q, \Delta q\right) = \tau_{\mathrm{q}}\left(q, \epsilon_{\mathrm{q},n}\left(\Delta q\right)\right)$$

Theorem 490 *If*

$$\psi = \left[\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}\right]$$

is a pitch system, $n_1, n_2, \ldots, n_k$ is a collection of integers, q is a chromamorph in ψ and Δq is a chromamorph interval in ψ then

$$\tau_{\mathrm{q},n_k}\left(\ldots\tau_{\mathrm{q},n_2}\left(\tau_{\mathrm{q},n_1}\left(q, \Delta q\right), \Delta q\right)\ldots, \Delta q\right) = \tau_{\mathrm{q},\sum_{j=1}^{k} n_j}\left(q, \Delta q\right)$$

Proof

R1	Let		$y_k = \tau_{\mathrm{q},n_k}\left(\ldots\tau_{\mathrm{q},n_2}\left(\tau_{\mathrm{q},n_1}\left(q,\Delta q\right),\Delta q\right)\ldots,\Delta q\right)$
R2	Let		$x_k = \tau_{\mathrm{q},\sum_{j=1}^{k} n_j}\left(q,\Delta q\right)$
R3	R1	$\Rightarrow$	$y_1 = \tau_{\mathrm{q},n_1}\left(q,\Delta q\right)$
R4	R2	$\Rightarrow$	$x_1 = \tau_{\mathrm{q},\sum_{j=1}^{1} n_j}\left(q,\Delta q\right)$
R5			$\sum_{j=1}^{1} n_j = n_1$
R6	R3, R4 & R5	$\Rightarrow$	$y_1 = x_1$
R7	R1 & R2	$\Rightarrow$	$\left(y_k = x_k \Rightarrow y_{k+1} = \tau_{\mathrm{q},n_{k+1}}\left(x_k,\Delta q\right)\right)$
R8	R2	$\Rightarrow$	$\tau_{\mathrm{q},n_{k+1}}\left(x_k,\Delta q\right) = \tau_{\mathrm{q},n_{k+1}}\left(\tau_{\mathrm{q},\sum_{j=1}^{k} n_j}\left(q,\Delta q\right),\Delta q\right)$
R9	R8 & 489	$\Rightarrow$	$\tau_{\mathrm{q},n_{k+1}}\left(x_k,\Delta q\right) = \tau_{\mathrm{q},n_{k+1}}\left(\tau_{\mathrm{q}}\left(q,\epsilon_{\mathrm{q},\sum_{j=1}^{k} n_j}\left(\Delta q\right)\right),\Delta q\right)$ $= \tau_{\mathrm{q}}\left(\tau_{\mathrm{q}}\left(q,\epsilon_{\mathrm{q},\sum_{j=1}^{k} n_j}\left(\Delta q\right)\right),\epsilon_{\mathrm{q},n_{k+1}}\left(\Delta q\right)\right)$
R10	476 & R9	$\Rightarrow$	$\tau_{\mathrm{q},n_{k+1}}\left(x_k,\Delta q\right) = \tau_{\mathrm{q}}\left(q,\sigma_{\mathrm{q}}\left(\epsilon_{\mathrm{q},\sum_{j=1}^{k} n_j}\left(\Delta q\right),\epsilon_{\mathrm{q},n_{k+1}}\left(\Delta q\right)\right)\right)$
R11	488 & R10	$\Rightarrow$	$\tau_{\mathrm{q},n_{k+1}}\left(x_k,\Delta q\right) = \tau_{\mathrm{q}}\left(q,\epsilon_{\mathrm{q},\left(\sum_{j=1}^{k} n_j\right)+n_{k+1}}\left(\Delta q\right)\right) = \tau_{\mathrm{q}}\left(q,\epsilon_{\mathrm{q},\sum_{j=1}^{k+1} n_j}\left(\Delta q\right)\right)$
R12	R2, R11 & 489	$\Rightarrow$	$\tau_{\mathrm{q},n_{k+1}}\left(x_k,\Delta q\right) = \tau_{\mathrm{q},\sum_{j=1}^{k+1} n_j}\left(q,\Delta q\right) = x_{k+1}$
R13	R7 & R12	$\Rightarrow$	$\left(y_k = x_k \Rightarrow y_{k+1} = x_{k+1}\right)$
R14	R13 & R6	$\Rightarrow$	$y_k = x_k$ for all integer k greater than zero.
R15	R14, R1 & R2	$\Rightarrow$	$\tau_{\mathrm{q},n_k}\left(\ldots\tau_{\mathrm{q},n_2}\left(\tau_{\mathrm{q},n_1}\left(q,\Delta q\right),\Delta q\right)\ldots,\Delta q\right) = \tau_{\mathrm{q},\sum_{j=1}^{k} n_j}\left(q,\Delta q\right)$

4.6.4 Summation, inversion and exponentiation of genus intervals

Summation of genus intervals

Definition 491 (Summation of genus intervals) *If*

$$\psi = [\mu_{\mathrm{c}},\mu_{\mathrm{m}},f_0,p_{\mathrm{c},0}]$$

is a pitch system and

$$\Delta g_1,\Delta g_2,\ldots\Delta g_n$$

is a collection of genus intervals in ψ then

$$\sigma_{\mathrm{g}}\left(\Delta g_1, \Delta g_2, \ldots \Delta g_n\right) = \left[\left(\sum_{k=1}^{n} \Delta\, \mathrm{g_c}\left(\Delta g_k\right)\right) - \mu_{\mathrm{c}} \times \left(\left(\sum_{k=1}^{n} \Delta\, \mathrm{m}\left(\Delta g_k\right)\right) \operatorname{div} \mu_{\mathrm{m}}\right), \left(\sum_{k=1}^{n} \Delta\, \mathrm{m}\left(\Delta g_k\right)\right) \bmod \mu_{\mathrm{m}}\right]$$

Theorem 492 *If*

$$\psi = \left[\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}\right]$$

is a pitch system, g is a genus in ψ and

$$\Delta g_1, \Delta g_2, \ldots \Delta g_n$$

is a collection of genus intervals in ψ then

$$\tau_{\mathrm{g}}\left(g, \sigma_{\mathrm{g}}\left(\Delta g_1, \Delta g_2, \ldots \Delta g_n\right)\right) = \left[\begin{array}{l} \mathrm{g_c}\left(g\right) + \left(\sum_{k=1}^{n} \Delta\, \mathrm{g_c}\left(\Delta g_k\right)\right) - \mu_{\mathrm{c}} \times \left(\left(\left(\sum_{k=1}^{n} \Delta\, \mathrm{m}\left(\Delta g_k\right)\right) + \mathrm{m}\left(g\right)\right) \operatorname{div} \mu_{\mathrm{m}}\right), \\ \\ \left(\mathrm{m}\left(g\right) + \left(\sum_{k=1}^{n} \Delta\, \mathrm{m}\left(\Delta g_k\right)\right)\right) \bmod \mu_{\mathrm{m}} \end{array}\right]$$

Proof

R1 491 & 422 $\Rightarrow$ $\tau_{\mathrm{g}}\left(g, \sigma_{\mathrm{g}}\left(\Delta g_1, \Delta g_2, \ldots \Delta g_n\right)\right)$

$$= \tau_{\mathrm{g}}\left(g, \begin{bmatrix} \left(\sum_{k=1}^{n} \Delta\,\mathrm{g_c}\left(\Delta g_k\right)\right) - \mu_{\mathrm{c}} \times \left(\left(\sum_{k=1}^{n} \Delta\,\mathrm{m}\left(\Delta g_k\right)\right) \operatorname{div} \mu_{\mathrm{m}}\right), \\ \left(\sum_{k=1}^{n} \Delta\,\mathrm{m}\left(\Delta g_k\right)\right) \bmod \mu_{\mathrm{m}} \end{bmatrix}\right)$$

$$= \begin{bmatrix} \mathrm{g_c}\left(g\right) + \left(\sum_{k=1}^{n} \Delta\,\mathrm{g_c}\left(\Delta g_k\right)\right) \\ -\mu_{\mathrm{c}} \times \left(\left(\sum_{k=1}^{n} \Delta\,\mathrm{m}\left(\Delta g_k\right)\right) \operatorname{div} \mu_{\mathrm{m}}\right) \\ -\mu_{\mathrm{c}} \times \left(\left(\mathrm{m}\left(g\right) + \left(\sum_{k=1}^{n} \Delta\,\mathrm{m}\left(\Delta g_k\right)\right) \bmod \mu_{\mathrm{m}}\right) \operatorname{div} \mu_{\mathrm{m}}\right), \\ \tau_{\mathrm{m}}\left(\mathrm{m}\left(g\right), \left(\sum_{k=1}^{n} \Delta\,\mathrm{m}\left(\Delta g_k\right)\right) \bmod \mu_{\mathrm{m}}\right) \end{bmatrix}$$

$$= \begin{bmatrix} \mathrm{g_c}\left(g\right) + \left(\sum_{k=1}^{n} \Delta\,\mathrm{g_c}\left(\Delta g_k\right)\right) \\ -\mu_{\mathrm{c}} \times \begin{pmatrix} \left(\left(\sum_{k=1}^{n} \Delta\,\mathrm{m}\left(\Delta g_k\right)\right) \operatorname{div} \mu_{\mathrm{m}}\right) \\ + \left(\left(\mathrm{m}\left(g\right) + \left(\sum_{k=1}^{n} \Delta\,\mathrm{m}\left(\Delta g_k\right)\right) \bmod \mu_{\mathrm{m}}\right) \operatorname{div} \mu_{\mathrm{m}}\right) \end{pmatrix}, \\ \tau_{\mathrm{m}}\left(\mathrm{m}\left(g\right), \left(\sum_{k=1}^{n} \Delta\,\mathrm{m}\left(\Delta g_k\right)\right) \bmod \mu_{\mathrm{m}}\right) \end{bmatrix}$$

R2 52 $\Rightarrow$ $\left(\left(\sum_{k=1}^{n} \Delta\,\mathrm{m}\left(\Delta g_k\right)\right) \operatorname{div} \mu_{\mathrm{m}}\right) + \left(\left(\mathrm{m}\left(g\right) + \left(\sum_{k=1}^{n} \Delta\,\mathrm{m}\left(\Delta g_k\right)\right) \bmod \mu_{\mathrm{m}}\right) \operatorname{div} \mu_{\mathrm{m}}\right)$

$$= \left(\left(\sum_{k=1}^{n} \Delta\,\mathrm{m}\left(\Delta g_k\right)\right) + \mathrm{m}\left(g\right)\right) \operatorname{div} \mu_{\mathrm{m}}$$

R3 R1 & R2 $\Rightarrow$ $\tau_{\mathrm{g}}\left(g, \sigma_{\mathrm{g}}\left(\Delta g_1, \Delta g_2, \ldots \Delta g_n\right)\right)$

$$= \begin{bmatrix} \mathrm{g_c}\left(g\right) + \left(\sum_{k=1}^{n} \Delta\,\mathrm{g_c}\left(\Delta g_k\right)\right) \\ -\mu_{\mathrm{c}} \times \left(\left(\left(\sum_{k=1}^{n} \Delta\,\mathrm{m}\left(\Delta g_k\right)\right) + \mathrm{m}\left(g\right)\right) \operatorname{div} \mu_{\mathrm{m}}\right), \\ \tau_{\mathrm{m}}\left(\mathrm{m}\left(g\right), \left(\sum_{k=1}^{n} \Delta\,\mathrm{m}\left(\Delta g_k\right)\right) \bmod \mu_{\mathrm{m}}\right) \end{bmatrix}$$

R4 R3 & 412 $\Rightarrow$ $\tau_{\mathrm{g}}\left(g, \sigma_{\mathrm{g}}\left(\Delta g_1, \Delta g_2, \ldots \Delta g_n\right)\right)$

$$= \begin{bmatrix} \mathrm{g_c}\left(g\right) + \left(\sum_{k=1}^{n} \Delta\,\mathrm{g_c}\left(\Delta g_k\right)\right) \\ -\mu_{\mathrm{c}} \times \left(\left(\left(\sum_{k=1}^{n} \Delta\,\mathrm{m}\left(\Delta g_k\right)\right) + \mathrm{m}\left(g\right)\right) \operatorname{div} \mu_{\mathrm{m}}\right), \\ \left(\left(\left(\mathrm{m}\left(g\right) + \left(\sum_{k=1}^{n} \Delta\,\mathrm{m}\left(\Delta g_k\right)\right)\right) \bmod \mu_{\mathrm{m}}\right) \bmod \mu_{\mathrm{m}}\right) \end{bmatrix}$$

R5 R4 & 35 $\Rightarrow$ $\tau_{\mathrm{g}}\left(g, \sigma_{\mathrm{g}}\left(\Delta g_1, \Delta g_2, \ldots \Delta g_n\right)\right) = \begin{bmatrix} \mathrm{g_c}\left(g\right) + \left(\sum_{k=1}^{n} \Delta\,\mathrm{g_c}\left(\Delta g_k\right)\right) \\ -\mu_{\mathrm{c}} \times \left(\left(\left(\sum_{k=1}^{n} \Delta\,\mathrm{m}\left(\Delta g_k\right)\right) + \mathrm{m}\left(g\right)\right) \operatorname{div} \mu_{\mathrm{m}}\right), \\ \left(\mathrm{m}\left(g\right) + \left(\sum_{k=1}^{n} \Delta\,\mathrm{m}\left(\Delta g_k\right)\right)\right) \bmod \mu_{\mathrm{m}} \end{bmatrix}$

Theorem 493 *If ψ is a pitch system and*

$$\Delta g_1, \Delta g_2, \ldots \Delta g_n$$

is a collection of genus intervals in ψ and g is a genus in ψ then

$$\tau_{\mathrm{g}}\left(g, \sigma_{\mathrm{g}}\left(\Delta g_1, \Delta g_2, \ldots \Delta g_n\right)\right) = \tau_{\mathrm{g}}\left(\ldots \tau_{\mathrm{g}}\left(\tau_{\mathrm{g}}\left(g, \Delta g_1\right), \Delta g_2\right) \ldots, \Delta g_n\right)$$

Proof

R1 Let $x_k = \tau_{\mathrm{g}}\left(g, \sigma_{\mathrm{g}}\left(\Delta g_1, \Delta g_2, \ldots \Delta g_k\right)\right)$

R2 Let $y_k = \tau_{\mathrm{g}}\left(\ldots \tau_{\mathrm{g}}\left(\tau_{\mathrm{g}}\left(g, \Delta g_1\right), \Delta g_2\right) \ldots, \Delta g_k\right)$

R3 R1 & 492 $\Rightarrow$ $x_1 = \tau_{\mathrm{g}}\left(g, \sigma_{\mathrm{g}}\left(\Delta g_1\right)\right)$

$$= \begin{bmatrix} \mathrm{g_c}\left(g\right) + \sum_{j=1}^{1} \Delta\, \mathrm{g_c}\left(\Delta g_j\right) \\ -\mu_{\mathrm{c}} \times \left(\left(\sum_{j=1}^{1} \Delta\, \mathrm{m}\left(\Delta g_j\right) + \mathrm{m}\left(g\right)\right) \mathrm{div}\ \mu_{\mathrm{m}}\right), \\ \left(\mathrm{m}\left(g\right) + \sum_{j=1}^{1} \Delta\, \mathrm{m}\left(\Delta g_j\right)\right) \mathrm{mod}\ \mu_{\mathrm{m}} \end{bmatrix}$$

$$= \begin{bmatrix} \mathrm{g_c}\left(g\right) + \Delta\, \mathrm{g_c}\left(\Delta g_1\right) \\ -\mu_{\mathrm{c}} \times \left(\left(\Delta\, \mathrm{m}\left(\Delta g_1\right) + \mathrm{m}\left(g\right)\right) \mathrm{div}\ \mu_{\mathrm{m}}\right), \\ \left(\mathrm{m}\left(g\right) + \Delta\, \mathrm{m}\left(\Delta g_1\right)\right) \mathrm{mod}\ \mu_{\mathrm{m}} \end{bmatrix}$$

R4 R2, 412 & 422 $\Rightarrow$ $y_1 = \tau_{\mathrm{g}}\left(g, \Delta g_1\right)$

$$= \begin{bmatrix} \mathrm{g_c}\left(g\right) + \Delta\, \mathrm{g_c}\left(\Delta g_1\right) \\ -\mu_{\mathrm{c}} \times \left(\left(\mathrm{m}\left(g\right) + \Delta\, \mathrm{m}\left(\Delta g_1\right)\right) \mathrm{div}\ \mu_{\mathrm{m}}\right), \\ \tau_{\mathrm{m}}\left(\mathrm{m}\left(g\right), \Delta\, \mathrm{m}\left(\Delta g_1\right)\right) \end{bmatrix}$$

$$= \begin{bmatrix} \mathrm{g_c}\left(g\right) + \Delta\, \mathrm{g_c}\left(\Delta g_1\right) \\ -\mu_{\mathrm{c}} \times \left(\left(\mathrm{m}\left(g\right) + \Delta\, \mathrm{m}\left(\Delta g_1\right)\right) \mathrm{div}\ \mu_{\mathrm{m}}\right), \\ \left(\mathrm{m}\left(g\right) + \Delta\, \mathrm{m}\left(\Delta g_1\right)\right) \mathrm{mod}\ \mu_{\mathrm{m}} \end{bmatrix}$$

R5 R3 & R4 $\Rightarrow$ $x_1 = y_1$

R6 R1 & R2 $\Rightarrow$ $\left(x_k = y_k \Rightarrow y_{k+1} = \tau_{\mathrm{g}}\left(x_k, \Delta g_{k+1}\right)\right)$

R7 R1 & 422 $\Rightarrow$ $\tau_{\mathrm{g}}\left(x_k, \Delta g_{k+1}\right) = \begin{bmatrix} \mathrm{g_c}\left(x_k\right) + \Delta\, \mathrm{g_c}\left(\Delta g_{k+1}\right) \\ -\mu_{\mathrm{c}} \times \left(\left(\mathrm{m}\left(x_k\right) + \Delta\, \mathrm{m}\left(\Delta g_{k+1}\right)\right) \mathrm{div}\ \mu_{\mathrm{m}}\right), \\ \tau_{\mathrm{m}}\left(\mathrm{m}\left(x_k\right), \Delta\, \mathrm{m}\left(\Delta g_{k+1}\right)\right) \end{bmatrix}$

R8 R7 & 412 $\Rightarrow$ $\tau_{\mathrm{g}}\left(x_k, \Delta g_{k+1}\right) = \begin{bmatrix} \mathrm{g_c}\left(x_k\right) + \Delta\, \mathrm{g_c}\left(\Delta g_{k+1}\right) \\ -\mu_{\mathrm{c}} \times \left(\left(\mathrm{m}\left(x_k\right) + \Delta\, \mathrm{m}\left(\Delta g_{k+1}\right)\right) \mathrm{div}\ \mu_{\mathrm{m}}\right), \\ \left(\mathrm{m}\left(x_k\right) + \Delta\, \mathrm{m}\left(\Delta g_{k+1}\right)\right) \mathrm{mod}\ \mu_{\mathrm{m}} \end{bmatrix}$

R9 R1 & 492 $\Rightarrow$ $$x_k = \begin{bmatrix} \mathrm{g_c}(g) + \sum_{j=1}^{k} \Delta\,\mathrm{g_c}(\Delta g_j) \\ -\mu_c \times \left(\left(\sum_{j=1}^{k} \Delta\,\mathrm{m}(\Delta g_j) + \mathrm{m}(g)\right) \text{ div } \mu_m\right), \\ \left(\mathrm{m}(g) + \sum_{j=1}^{k} \Delta\,\mathrm{m}(\Delta g_j)\right) \text{ mod } \mu_m \end{bmatrix}$$

R10 R8, R9, 115 & 117 $\Rightarrow$ $\tau_g(x_k, \Delta g_{k+1})$

$$= \begin{bmatrix} \mathrm{g_c}(g) + \sum_{j=1}^{k} \Delta\,\mathrm{g_c}(\Delta g_j) \\ -\mu_c \times \left(\left(\sum_{j=1}^{k} \Delta\,\mathrm{m}(\Delta g_j) + \mathrm{m}(g)\right) \text{ div } \mu_m\right) \\ +\Delta\,\mathrm{g_c}(\Delta g_{k+1}) \\ -\mu_c \times \left(\left(\left(\mathrm{m}(g) + \sum_{j=1}^{k} \Delta\,\mathrm{m}(\Delta g_j)\right) \text{ mod } \mu_m + \Delta\,\mathrm{m}(\Delta g_{k+1})\right) \text{ div } \mu_m\right), \\ \left(\left(\mathrm{m}(g) + \sum_{j=1}^{k} \Delta\,\mathrm{m}(\Delta g_j)\right) \text{ mod } \mu_m + \Delta\,\mathrm{m}(\Delta g_{k+1})\right) \text{ mod } \mu_m \end{bmatrix}$$

$$= \begin{bmatrix} \mathrm{g_c}(g) + \sum_{j=1}^{k+1} \Delta\,\mathrm{g_c}(\Delta g_j) \\ -\mu_c \times \begin{pmatrix} \left(\sum_{j=1}^{k} \Delta\,\mathrm{m}(\Delta g_j) + \mathrm{m}(g)\right) \text{ div } \mu_m \\ + \left(\Delta\,\mathrm{m}(\Delta g_{k+1}) + \left(\mathrm{m}(g) + \sum_{j=1}^{k} \Delta\,\mathrm{m}(\Delta g_j)\right) \text{ mod } \mu_m\right) \text{ div } \mu_m \end{pmatrix}, \\ \left(\left(\mathrm{m}(g) + \sum_{j=1}^{k} \Delta\,\mathrm{m}(\Delta g_j)\right) \text{ mod } \mu_m + \Delta\,\mathrm{m}(\Delta g_{k+1})\right) \text{ mod } \mu_m \end{bmatrix}$$

R11 R9 $\Rightarrow$ $$x_{k+1} = \begin{bmatrix} \mathrm{g_c}(g) + \sum_{j=1}^{k+1} \Delta\,\mathrm{g_c}(\Delta g_j) \\ -\mu_c \times \left(\left(\sum_{j=1}^{k+1} \Delta\,\mathrm{m}(\Delta g_j) + \mathrm{m}(g)\right) \text{ div } \mu_m\right), \\ \left(\mathrm{m}(g) + \sum_{j=1}^{k+1} \Delta\,\mathrm{m}(\Delta g_j)\right) \text{ mod } \mu_m \end{bmatrix}$$

R12 Let $$w_k = \left(\sum_{j=1}^{k} \Delta\,\mathrm{m}(\Delta g_j) + \mathrm{m}(g)\right) \text{ div } \mu_m$$

$$+ \left(\Delta\,\mathrm{m}(\Delta g_{k+1}) + \left(\mathrm{m}(g) + \sum_{j=1}^{k} \Delta\,\mathrm{m}(\Delta g_j)\right) \text{ mod } \mu_m\right) \text{ div } \mu_m$$

R13 R12 & 52 $\Rightarrow$ $$w_k = \left(\Delta\,\mathrm{m}(\Delta g_{k+1}) + \sum_{j=1}^{k} \Delta\,\mathrm{m}(\Delta g_j) + \mathrm{m}(g)\right) \text{ div } \mu_m$$

$$= \left(\sum_{j=1}^{k+1} \Delta\,\mathrm{m}(\Delta g_j) + \mathrm{m}(g)\right) \text{ div } \mu_m$$

R14 Let $$z_k = \left(\left(\mathrm{m}(g) + \sum_{j=1}^{k} \Delta\,\mathrm{m}(\Delta g_j)\right) \text{ mod } \mu_m + \Delta\,\mathrm{m}(\Delta g_{k+1})\right) \text{ mod } \mu_m$$

R15 R14 & 38 $\Rightarrow$ $z_k = \left(\Delta \mathrm{m}(\Delta g_{k+1}) + \mathrm{m}(g) + \sum_{j=1}^{k} \Delta \mathrm{m}(\Delta g_j)\right) \bmod \mu_{\mathrm{m}}$

$= \left(\mathrm{m}(g) + \sum_{j=1}^{k+1} \Delta \mathrm{m}(\Delta g_j)\right) \bmod \mu_{\mathrm{m}}$

R16 R10, R12 & R14 $\Rightarrow$ $\tau_{\mathrm{g}}(x_k, \Delta g_{k+1}) = \left[\mathrm{g_c}(g) + \sum_{j=1}^{k+1} \Delta \mathrm{g_c}(\Delta g_j) - \mu_{\mathrm{c}} \times w_k, z_k\right]$

R17 R11, R13 & R15 $\Rightarrow$ $x_{k+1} = \left[\mathrm{g_c}(g) + \sum_{j=1}^{k+1} \Delta \mathrm{g_c}(\Delta g_j) - \mu_{\mathrm{c}} \times w_k, z_k\right]$

R18 R16 & R17 $\Rightarrow$ $\tau_{\mathrm{g}}(x_k, \Delta g_{k+1}) = x_{k+1}$

R19 R6 & R18 $\Rightarrow$ $(x_k = y_k \Rightarrow x_{k+1} = y_{k+1})$

R20 R19 & R5 $\Rightarrow$ $x_k = y_k$ for all integers k greater than zero.

R21 R20, R1 & R2 $\Rightarrow$ $\tau_{\mathrm{g}}(g, \sigma_{\mathrm{g}}(\Delta g_1, \Delta g_2, \ldots \Delta g_n)) = \tau_{\mathrm{g}}(\ldots \tau_{\mathrm{g}}(\tau_{\mathrm{g}}(g, \Delta g_1), \Delta g_2) \ldots, \Delta g_n)$

Inverse of a genus interval

Definition 494 (Inverse of a genus interval) *If ψ is a pitch system and Δg is a genus interval in ψ and g is a genus in ψ then the* inverse of Δg, *denoted $\iota_{\mathrm{g}}(\Delta g)$, is the genus interval that satisfies the following equation*

$$\tau_{\mathrm{g}}(\tau_{\mathrm{g}}(g, \Delta g), \iota_{\mathrm{g}}(\Delta g)) = g$$

Definition 495 (Inversional equivalence of genus intervals) *If ψ is a pitch system and Δg_1 and Δg_2 are genus intervals in ψ then Δg_1 and Δg_2 are* inversionally equivalent *if and only if*

$$(\iota_{\mathrm{g}}(\Delta g_1) = \Delta g_2) \vee (\Delta g_1 = \Delta g_2)$$

The fact that two genus intervals are inversionally equivalent is denoted as follows:

$$\Delta g_1 \equiv_\iota \Delta g_2$$

Theorem 496 *If*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

is a pitch system and Δg is a genus interval in ψ then

$$\iota_{\mathrm{g}}(\Delta g) = [\mu_{\mathrm{c}} - \Delta \mathrm{g_c}(\Delta g), (-\Delta \mathrm{m}(\Delta g)) \bmod \mu_{\mathrm{m}}]$$

Proof

R1 Let $x = \tau_{\mathrm{g}}\left(\tau_{\mathrm{g}}\left(g, \Delta g\right), \left[\mu_{\mathrm{c}} - \Delta\,\mathrm{g_c}\left(\Delta g\right), \left(-\Delta\,\mathrm{m}\left(\Delta g\right)\right) \bmod \mu_{\mathrm{m}}\right]\right)$

R2 R1, 422, 115, 117, 310 & 316 $\Rightarrow$
$$x = \tau_{\mathrm{g}}\left(\begin{array}{l}\left[\begin{array}{l}\mathrm{g_c}\left(g\right) + \Delta\,\mathrm{g_c}\left(\Delta g\right) - \mu_{\mathrm{c}} \times \left(\left(\mathrm{m}\left(g\right) + \Delta\,\mathrm{m}\left(\Delta g\right)\right) \operatorname{div} \mu_{\mathrm{m}}\right), \\ \tau_{\mathrm{m}}\left(\mathrm{m}\left(g\right), \Delta\,\mathrm{m}\left(\Delta g\right)\right)\end{array}\right], \\ \left[\mu_{\mathrm{c}} - \Delta\,\mathrm{g_c}\left(\Delta g\right), \left(-\Delta\,\mathrm{m}\left(\Delta g\right)\right) \bmod \mu_{\mathrm{m}}\right]\end{array}\right)$$
$$= \left[\begin{array}{l}\mathrm{g_c}\left(g\right) + \Delta\,\mathrm{g_c}\left(\Delta g\right) - \mu_{\mathrm{c}} \times \left(\left(\mathrm{m}\left(g\right) + \Delta\,\mathrm{m}\left(\Delta g\right)\right) \operatorname{div} \mu_{\mathrm{m}}\right) + \mu_{\mathrm{c}} - \Delta\,\mathrm{g_c}\left(\Delta g\right) \\ -\mu_{\mathrm{c}} \times \left(\left(\tau_{\mathrm{m}}\left(\mathrm{m}\left(g\right), \Delta\,\mathrm{m}\left(\Delta g\right)\right) + \left(-\Delta\,\mathrm{m}\left(\Delta g\right)\right) \bmod \mu_{\mathrm{m}}\right) \operatorname{div} \mu_{\mathrm{m}}\right), \\ \tau_{\mathrm{m}}\left(\tau_{\mathrm{m}}\left(\mathrm{m}\left(g\right), \Delta\,\mathrm{m}\left(\Delta g\right)\right), \left(-\Delta\,\mathrm{m}\left(\Delta g\right)\right) \bmod \mu_{\mathrm{m}}\right)\end{array}\right]$$

R3 R2 & 412 $\Rightarrow$
$$x = \left[\begin{array}{l}\mathrm{g_c}\left(g\right) + \Delta\,\mathrm{g_c}\left(\Delta g\right) - \mu_{\mathrm{c}} \times \left(\left(\mathrm{m}\left(g\right) + \Delta\,\mathrm{m}\left(\Delta g\right)\right) \operatorname{div} \mu_{\mathrm{m}}\right) + \mu_{\mathrm{c}} - \Delta\,\mathrm{g_c}\left(\Delta g\right) \\ -\mu_{\mathrm{c}} \times \left(\left(\left(\mathrm{m}\left(g\right) + \Delta\,\mathrm{m}\left(\Delta g\right)\right) \bmod \mu_{\mathrm{m}} + \left(-\Delta\,\mathrm{m}\left(\Delta g\right)\right) \bmod \mu_{\mathrm{m}}\right) \operatorname{div} \mu_{\mathrm{m}}\right), \\ \left(\left(\mathrm{m}\left(g\right) + \Delta\,\mathrm{m}\left(\Delta g\right)\right) \bmod \mu_{\mathrm{m}} + \left(-\Delta\,\mathrm{m}\left(\Delta g\right)\right) \bmod \mu_{\mathrm{m}}\right) \bmod \mu_{\mathrm{m}}\end{array}\right]$$
$$= \left[\begin{array}{l}\mathrm{g_c}\left(g\right) + \mu_{\mathrm{c}} \\ -\mu_{\mathrm{c}} \times \left(\begin{array}{l}\left(\mathrm{m}\left(g\right) + \Delta\,\mathrm{m}\left(\Delta g\right)\right) \operatorname{div} \mu_{\mathrm{m}} \\ + \left(\left(\mathrm{m}\left(g\right) + \Delta\,\mathrm{m}\left(\Delta g\right)\right) \bmod \mu_{\mathrm{m}} + \left(-\Delta\,\mathrm{m}\left(\Delta g\right)\right) \bmod \mu_{\mathrm{m}}\right) \operatorname{div} \mu_{\mathrm{m}}\end{array}\right), \\ \left(\begin{array}{l}\left(\mathrm{m}\left(g\right) + \Delta\,\mathrm{m}\left(\Delta g\right)\right) \bmod \mu_{\mathrm{m}} \\ + \left(-\Delta\,\mathrm{m}\left(\Delta g\right)\right) \bmod \mu_{\mathrm{m}}\end{array}\right) \bmod \mu_{\mathrm{m}}\end{array}\right]$$

R4 R3, 52 & 34 $\Rightarrow$
$$x = \left[\begin{array}{l}\mathrm{g_c}\left(g\right) + \mu_{\mathrm{c}} - \mu_{\mathrm{c}} \times \left(\left(\mathrm{m}\left(g\right) + \Delta\,\mathrm{m}\left(\Delta g\right) + \left(-\Delta\,\mathrm{m}\left(\Delta g\right)\right) \bmod \mu_{\mathrm{m}}\right) \operatorname{div} \mu_{\mathrm{m}}\right), \\ \left(\mathrm{m}\left(g\right) + \Delta\,\mathrm{m}\left(\Delta g\right) - \Delta\,\mathrm{m}\left(\Delta g\right)\right) \bmod \mu_{\mathrm{m}}\end{array}\right]$$

R5 R4 & 46 $\Rightarrow$ $x = \left[\mathrm{g_c}\left(g\right) + \mu_{\mathrm{c}} - \mu_{\mathrm{c}} \times \left(\left(\mathrm{m}\left(g\right) + \mu_{\mathrm{m}}\right) \operatorname{div} \mu_{\mathrm{m}}\right), \mathrm{m}\left(g\right) \bmod \mu_{\mathrm{m}}\right]$

R6 77 & 61 $\Rightarrow$ $\left(\mathrm{m}\left(g\right) + \mu_{\mathrm{m}}\right) \operatorname{div} \mu_{\mathrm{m}} = 1$

R7 77 & 44 $\Rightarrow$ $\mathrm{m}\left(g\right) \bmod \mu_{\mathrm{m}} = \mathrm{m}\left(g\right)$

R8 R5, R6 & R7 $\Rightarrow$ $x = \left[\mathrm{g_c}\left(g\right) + \mu_{\mathrm{c}} - \mu_{\mathrm{c}} \times 1, \mathrm{m}\left(g\right)\right]$

$= \left[\mathrm{g_c}\left(g\right), \mathrm{m}\left(g\right)\right]$

R9 R8 & 118 $\Rightarrow$ $x = g$

R10 R1, R9 & 494 $\Rightarrow$ $\iota_{\mathrm{g}}\left(\Delta g\right) = \left[\mu_{\mathrm{c}} - \Delta\,\mathrm{g_c}\left(\Delta g\right), \left(-\Delta\,\mathrm{m}\left(\Delta g\right)\right) \bmod \mu_{\mathrm{m}}\right]$

Theorem 497 *If ψ is a pitch system and Δg, Δg_1 and Δg_2 are genus intervals in ψ then*

$$(\Delta g_1 = \iota_{\mathrm{g}}(\Delta g)) \wedge (\Delta g_2 = \iota_{\mathrm{g}}(\Delta g)) \Rightarrow (\Delta g_1 = \Delta g_2)$$

Proof

R1	Let		$\Delta g_1 = \iota_{\mathrm{g}}(\Delta g)$
R2	Let		$\Delta g_2 = \iota_{\mathrm{g}}(\Delta g)$
R3	R1 & 494	$\Rightarrow$	$\tau_{\mathrm{g}}(\tau_{\mathrm{g}}(g, \Delta g), \Delta g_1) = g$
R4	R2 & 494	$\Rightarrow$	$\tau_{\mathrm{g}}(\tau_{\mathrm{g}}(g, \Delta g), \Delta g_2) = g$
R5	R3, R4 & 425	$\Rightarrow$	$\Delta g_1 = \Delta g_2$
R6	R1 to R5	$\Rightarrow$	$(\Delta g_1 = \iota_{\mathrm{g}}(\Delta g)) \wedge (\Delta g_2 = \iota_{\mathrm{g}}(\Delta g)) \Rightarrow (\Delta g_1 = \Delta g_2)$

Theorem 498 *If ψ is a pitch system and Δg_1 and Δg_2 are two intervals in ψ then*

$$(\Delta g_1 = \iota_{\mathrm{g}}(\Delta g_2)) \iff (\Delta g_2 = \iota_{\mathrm{g}}(\Delta g_1))$$

Proof

Theorem 499 *The inversional equivalence relation on genus intervals is transitive. That is, if Δg_1, Δg_2 and Δg_3 are any three genus intervals in a pitch system ψ, then*

$$(\Delta g_1 \equiv_\iota \Delta g_2) \wedge (\Delta g_2 \equiv_\iota \Delta g_3) \Rightarrow (\Delta g_1 \equiv_\iota \Delta g_3)$$

Exponentiation of a genus interval

Definition 500 (Exponentiation of a genus interval) *Given that:*

1. *ψ is a pitch system;*
2. *g is a genus in ψ;*
3. *Δg is a genus interval in ψ;*
4. *n is an integer;*
5. *k is an integer and $1 \leq k \leq \mathrm{abs}(n)$;*
6. *$\Delta g_{1,k} = \Delta g$ for all k; and*
7. *$\Delta g_{2,k} = \iota_{\mathrm{g}}(\Delta g)$ for all k;*

then $\epsilon_{\mathrm{g},n}(\Delta g)$ returns a genus interval that satisfies the following equation:

$$\tau_{\mathrm{g}}(g, \epsilon_{\mathrm{g},n}(\Delta g)) = \begin{cases} \tau_{\mathrm{g}}(g, \sigma_{\mathrm{g}}(\Delta g_{1,1}, \Delta g_{1,2}, \ldots \Delta g_{1,n})) & \text{if} \quad n > 0 \\ g & \text{if} \quad n = 0 \\ \tau_{\mathrm{g}}(g, \sigma_{\mathrm{g}}(\Delta g_{2,1}, \Delta g_{2,2}, \ldots \Delta g_{2,-n})) & \text{if} \quad n < 0 \end{cases}$$

Theorem 501 (Formula for $\epsilon_{\mathrm{g},n}(\Delta g)$) *If*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

is a pitch system and Δg is a genus interval in ψ and n is an integer then

$$\epsilon_{\mathrm{g},n}(\Delta g) = \begin{bmatrix} n \times \Delta\,\mathrm{g_c}(\Delta g) - \mu_{\mathrm{c}} \times ((n \times \Delta\,\mathrm{m}(\Delta g)) \operatorname{div} \mu_{\mathrm{m}}), \\ \\ (n \times \Delta\,\mathrm{m}(\Delta g)) \bmod \mu_{\mathrm{m}} \end{bmatrix}$$

Proof

R1	Let		n be any integer
R2	Let		k be any integer such that $1 \le k \le \mathrm{abs}\,(n)$
R3	Let		$\Delta g_{1,k} = \Delta g$ for all k
R4	Let		$\Delta g_{2,k} = \iota_{\mathrm{g}}\,(\Delta g)$ for all k
R5	R1 to R4 & 500	$\Rightarrow$	$\tau_{\mathrm{g}}\,(g, \epsilon_{\mathrm{g},n}\,(\Delta g)) = \begin{cases} \tau_{\mathrm{g}}\,(g, \sigma_{\mathrm{g}}\,(\Delta g_{1,1}, \Delta g_{1,2}, \ldots \Delta g_{1,n})) & \text{if} \quad n > 0 \\ g & \text{if} \quad n = 0 \\ \tau_{\mathrm{g}}\,(g, \sigma_{\mathrm{g}}\,(\Delta g_{2,1}, \Delta g_{2,2}, \ldots \Delta g_{2,-n})) & \text{if} \quad n < 0 \end{cases}$
R6	Let		n_1 be any integer greater than zero
R7	491 & R6	$\Rightarrow$	$\sigma_{\mathrm{g}}\,(\Delta g_{1,1}, \Delta g_{1,2}, \ldots \Delta g_{1,n_1})$
			$= \begin{bmatrix} (\sum_{k=1}^{n_1} \Delta\,\mathrm{g_c}\,(\Delta g_{1,k})) - \mu_{\mathrm{c}} \times ((\sum_{k=1}^{n_1} \Delta\,\mathrm{m}\,(\Delta g_{1,k}))\ \mathrm{div}\ \mu_{\mathrm{m}})\,, \\ (\sum_{k=1}^{n_1} \Delta\,\mathrm{m}\,(\Delta g_{1,k}))\ \mathrm{mod}\ \mu_{\mathrm{m}} \end{bmatrix}$
R8	R3 & R7	$\Rightarrow$	$\sigma_{\mathrm{g}}\,(\Delta g_{1,1}, \Delta g_{1,2}, \ldots \Delta g_{1,n_1})$
			$= \begin{bmatrix} n_1 \times \Delta\,\mathrm{g_c}\,(\Delta g) - \mu_{\mathrm{c}} \times ((n_1 \times \Delta\,\mathrm{m}\,(\Delta g))\ \mathrm{div}\ \mu_{\mathrm{m}})\,, \\ (n_1 \times \Delta\,\mathrm{m}\,(\Delta g))\ \mathrm{mod}\ \mu_{\mathrm{m}} \end{bmatrix}$
R9	R1, R6 & R8	$\Rightarrow$	$\tau_{\mathrm{g}}\,(g, \sigma_{\mathrm{g}}\,(\Delta g_{1,1}, \Delta g_{1,2}, \ldots \Delta g_{1,n}))$
			$= \tau_{\mathrm{g}} \left(g, \begin{bmatrix} n \times \Delta\,\mathrm{g_c}\,(\Delta g) - \mu_{\mathrm{c}} \times ((n \times \Delta\,\mathrm{m}\,(\Delta g))\ \mathrm{div}\ \mu_{\mathrm{m}})\,, \\ (n \times \Delta\,\mathrm{m}\,(\Delta g))\ \mathrm{mod}\ \mu_{\mathrm{m}} \end{bmatrix} \right)$ when $n > 0$
R10	Let		$n_2 = 0$
R11	Let		$x = \tau_{\mathrm{g}} \left(g, \begin{bmatrix} n_2 \times \Delta\,\mathrm{g_c}\,(\Delta g) - \mu_{\mathrm{c}} \times ((n_2 \times \Delta\,\mathrm{m}\,(\Delta g))\ \mathrm{div}\ \mu_{\mathrm{m}})\,, \\ (n_2 \times \Delta\,\mathrm{m}\,(\Delta g))\ \mathrm{mod}\ \mu_{\mathrm{m}} \end{bmatrix} \right)$
R12	R10, R11, 422, 310 & 316	$\Rightarrow$	$x = \tau_{\mathrm{g}}\,(g, [0 - \mu_{\mathrm{c}} \times 0, 0]) = \tau_{\mathrm{g}}\,(g, [0, 0])$
			$= [\mathrm{g_c}\,(g) + 0 - \mu_{\mathrm{c}} \times ((\mathrm{m}\,(g) + 0)\ \mathrm{div}\ \mu_{\mathrm{m}})\,, \tau_{\mathrm{m}}\,(\mathrm{m}\,(g)\,, 0)]$
R13	R11, R12 & 412	$\Rightarrow$	$x = [\mathrm{g_c}\,(g) - \mu_{\mathrm{c}} \times (\mathrm{m}\,(g)\ \mathrm{div}\ \mu_{\mathrm{m}})\,, (\mathrm{m}\,(g) + 0)\ \mathrm{mod}\ \mu_{\mathrm{m}}]$
R14	R13 & 78	$\Rightarrow$	$x = [\mathrm{g_c}\,(g) - \mu_{\mathrm{c}} \times (\mathrm{m}\,(g)\ \mathrm{div}\ \mu_{\mathrm{m}})\,, \mathrm{m}\,(g)]$
R15	R14 & 79	$\Rightarrow$	$x = [\mathrm{g_c}\,(g) - \mu_{\mathrm{c}} \times 0, \mathrm{m}\,(g)] = [\mathrm{g_c}\,(g)\,, \mathrm{m}\,(g)]$

R16 R15 & 118 $\Rightarrow$ $x = g$

R17 R1, R10, R11 & R16 $\Rightarrow$ $\tau_{\mathrm{g}}\left(g, \begin{bmatrix} n \times \Delta\,\mathrm{g_c}\,(\Delta g) - \mu_{\mathrm{c}} \times ((n \times \Delta\,\mathrm{m}\,(\Delta g))\ \mathrm{div}\ \mu_{\mathrm{m}})\,, \\ \\ (n \times \Delta\,\mathrm{m}\,(\Delta g))\ \mathrm{mod}\ \mu_{\mathrm{m}} \end{bmatrix}\right) = g$ when $n = 0$

R18 Let n_3 be any integer less than zero

R19 Let $y = \sigma_{\mathrm{g}}\,(\Delta g_{2,1}, \Delta g_{2,2}, \ldots \Delta g_{2,-n_3})$

R20 R19 & 491 $\Rightarrow$ $y = \begin{bmatrix} \left(\sum_{k=1}^{-n_3} \Delta\,\mathrm{g_c}\,(\Delta g_{2,k})\right) - \mu_{\mathrm{c}} \times \left(\left(\sum_{k=1}^{-n_3} \Delta\,\mathrm{m}\,(\Delta g_{2,k})\right)\ \mathrm{div}\ \mu_{\mathrm{m}}\right), \\ \\ \left(\sum_{k=1}^{-n_3} \Delta\,\mathrm{m}\,(\Delta g_{2,k})\right)\ \mathrm{mod}\ \mu_{\mathrm{m}} \end{bmatrix}$

R21 R4 & R20 $\Rightarrow$ $y = \begin{bmatrix} -n_3 \times \Delta\,\mathrm{g_c}\,(\iota_{\mathrm{g}}\,(\Delta g)) - \mu_{\mathrm{c}} \times ((-n_3 \times \Delta\,\mathrm{m}\,(\iota_{\mathrm{g}}\,(\Delta g)))\ \mathrm{div}\ \mu_{\mathrm{m}})\,, \\ \\ (-n_3 \times \Delta\,\mathrm{m}\,(\iota_{\mathrm{g}}\,(\Delta g)))\ \mathrm{mod}\ \mu_{\mathrm{m}} \end{bmatrix}$

R22 R21, 310, 316 & 496 $\Rightarrow$ $y = \begin{bmatrix} -n_3 \times (\mu_{\mathrm{c}} - \Delta\,\mathrm{g_c}\,(\Delta g)) - \mu_{\mathrm{c}} \times ((-n_3 \times ((-\Delta\,\mathrm{m}\,(\Delta g))\ \mathrm{mod}\ \mu_{\mathrm{m}}))\ \mathrm{div}\ \mu_{\mathrm{m}})\,, \\ \\ (-n_3 \times ((-\Delta\,\mathrm{m}\,(\Delta g))\ \mathrm{mod}\ \mu_{\mathrm{m}}))\ \mathrm{mod}\ \mu_{\mathrm{m}} \end{bmatrix}$

$= \begin{bmatrix} n_3 \times \Delta\,\mathrm{g_c}\,(\Delta g) - \mu_{\mathrm{c}} \times (n_3 + (-n_3 \times ((-\Delta\,\mathrm{m}\,(\Delta g))\ \mathrm{mod}\ \mu_{\mathrm{m}}))\ \mathrm{div}\ \mu_{\mathrm{m}})\,, \\ \\ (-n_3 \times ((-\Delta\,\mathrm{m}\,(\Delta g))\ \mathrm{mod}\ \mu_{\mathrm{m}}))\ \mathrm{mod}\ \mu_{\mathrm{m}} \end{bmatrix}$

R23 218 & 45 $\Rightarrow$ $(-n_3 \times ((-\Delta\,\mathrm{m}\,(\Delta g))\ \mathrm{mod}\ \mu_{\mathrm{m}}))\ \mathrm{mod}\ \mu_{\mathrm{m}}$

$= (-n_3 \times (-\Delta\,\mathrm{m}\,(\Delta g)))\ \mathrm{mod}\ \mu_{\mathrm{m}}$

$= (n_3 \times \Delta\,\mathrm{m}\,(\Delta g))\ \mathrm{mod}\ \mu_{\mathrm{m}}$

R24 R22 & R23 $\Rightarrow$ $y = \begin{bmatrix} n_3 \times \Delta\,\mathrm{g_c}\,(\Delta g) - \mu_{\mathrm{c}} \times (n_3 + (-n_3 \times ((-\Delta\,\mathrm{m}\,(\Delta g))\ \mathrm{mod}\ \mu_{\mathrm{m}}))\ \mathrm{div}\ \mu_{\mathrm{m}})\,, \\ \\ (n_3 \times \Delta\,\mathrm{m}\,(\Delta g))\ \mathrm{mod}\ \mu_{\mathrm{m}} \end{bmatrix}$

R25 56 $\Rightarrow$ $n_3 + (-n_3 \times ((-\Delta\,\mathrm{m}\,(\Delta g))\ \mathrm{mod}\ \mu_{\mathrm{m}}))\ \mathrm{div}\ \mu_{\mathrm{m}} = (n_3 \times \Delta\,\mathrm{m}\,(\Delta g))\ \mathrm{div}\ \mu_{\mathrm{m}}$

R26 R24 & R25 $\Rightarrow$ $y = \begin{bmatrix} n_3 \times \Delta\,\mathrm{g_c}\,(\Delta g) - \mu_{\mathrm{c}} \times ((n_3 \times \Delta\,\mathrm{m}\,(\Delta g))\ \mathrm{div}\ \mu_{\mathrm{m}})\,, \\ \\ (n_3 \times \Delta\,\mathrm{m}\,(\Delta g))\ \mathrm{mod}\ \mu_{\mathrm{m}} \end{bmatrix}$

R27 R1, R18, R19 & R26 $\Rightarrow$ $\tau_{\mathrm{g}}\,(g, \sigma_{\mathrm{g}}\,(\Delta g_{2,1}, \Delta g_{2,2}, \ldots \Delta g_{2,-n}))$

$= \tau_{\mathrm{g}}\left(g, \begin{bmatrix} n \times \Delta\,\mathrm{g_c}\,(\Delta g) - \mu_{\mathrm{c}} \times ((n \times \Delta\,\mathrm{m}\,(\Delta g))\ \mathrm{div}\ \mu_{\mathrm{m}})\,, \\ \\ (n \times \Delta\,\mathrm{m}\,(\Delta g))\ \mathrm{mod}\ \mu_{\mathrm{m}} \end{bmatrix}\right)$ when $n < 0$

R28 R5, R9, R17 & R27 $\Rightarrow$ $\tau_{\mathrm{g}}\left(g, \epsilon_{\mathrm{g},n}\left(\Delta g\right)\right)$

$$= \tau_{\mathrm{g}}\left(g, \begin{bmatrix} n \times \Delta\,\mathrm{g_c}\left(\Delta g\right) - \mu_{\mathrm{c}} \times \left(\left(n \times \Delta\,\mathrm{m}\left(\Delta g\right)\right) \operatorname{div} \mu_{\mathrm{m}}\right), \\ \\ \left(n \times \Delta\,\mathrm{m}\left(\Delta g\right)\right) \bmod \mu_{\mathrm{m}} \end{bmatrix}\right) \text{ for all integer } n$$

R29 R28 & 425 $\Rightarrow$ $\epsilon_{\mathrm{g},n}\left(\Delta g\right) = \begin{bmatrix} n \times \Delta\,\mathrm{g_c}\left(\Delta g\right) - \mu_{\mathrm{c}} \times \left(\left(n \times \Delta\,\mathrm{m}\left(\Delta g\right)\right) \operatorname{div} \mu_{\mathrm{m}}\right), \\ \\ \left(n \times \Delta\,\mathrm{m}\left(\Delta g\right)\right) \bmod \mu_{\mathrm{m}} \end{bmatrix}$

Theorem 502 *If*

$$\psi = \left[\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}\right]$$

is a pitch system and Δg is any genus interval in ψ then

$$\iota_{\mathrm{g}}\left(\Delta g\right) = \epsilon_{\mathrm{g},-1}\left(\Delta g\right)$$

Proof

R1 496 $\Rightarrow$ $\iota_{\mathrm{g}}\left(\Delta g\right) = \left[\mu_{\mathrm{c}} - \Delta\,\mathrm{g_c}\left(\Delta g\right), \left(-\Delta\,\mathrm{m}\left(\Delta g\right)\right) \bmod \mu_{\mathrm{m}}\right]$

R2 501 $\Rightarrow$ $\epsilon_{\mathrm{g},-1}\left(\Delta g\right) = \begin{bmatrix} -1 \times \Delta\,\mathrm{g_c}\left(\Delta g\right) - \mu_{\mathrm{c}} \times \left(\left(-1 \times \Delta\,\mathrm{m}\left(\Delta g\right)\right) \operatorname{div} \mu_{\mathrm{m}}\right), \\ \\ \left(-1 \times \Delta\,\mathrm{m}\left(\Delta g\right)\right) \bmod \mu_{\mathrm{m}} \end{bmatrix}$

$$= \begin{bmatrix} -\Delta\,\mathrm{g_c}\left(\Delta g\right) - \mu_{\mathrm{c}} \times \left(\left(-\Delta\,\mathrm{m}\left(\Delta g\right)\right) \operatorname{div} \mu_{\mathrm{m}}\right), \\ \\ \left(-\Delta\,\mathrm{m}\left(\Delta g\right)\right) \bmod \mu_{\mathrm{m}} \end{bmatrix}$$

R3 218 $\Rightarrow$ $\left(-\Delta\,\mathrm{m}\left(\Delta g\right)\right) \operatorname{div} \mu_{\mathrm{m}} = -1$

R4 R2 & R3 $\Rightarrow$ $\epsilon_{\mathrm{g},-1}\left(\Delta g\right) = \begin{bmatrix} -\Delta\,\mathrm{g_c}\left(\Delta g\right) - \mu_{\mathrm{c}} \times \left(-1\right), \\ \\ \left(-\Delta\,\mathrm{m}\left(\Delta g\right)\right) \bmod \mu_{\mathrm{m}} \end{bmatrix}$

$$= \left[\mu_{\mathrm{c}} - \Delta\,\mathrm{g_c}\left(\Delta g\right), \left(-\Delta\,\mathrm{m}\left(\Delta g\right)\right) \bmod \mu_{\mathrm{m}}\right]$$

R5 R4 & R1 $\Rightarrow$ $\iota_{\mathrm{g}}\left(\Delta g\right) = \epsilon_{\mathrm{g},-1}\left(\Delta g\right)$

Theorem 503 *If*

$$\psi = \left[\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}\right]$$

is a pitch system, $n_1, n_2, \ldots n_k$ is a collection of integers and Δg is a genus interval in ψ then

$$\epsilon_{\mathrm{g},n_k}\left(\ldots \epsilon_{\mathrm{g},n_2}\left(\epsilon_{\mathrm{g},n_1}\left(\Delta g\right)\right)\ldots\right) = \epsilon_{\mathrm{g},\prod_{j=1}^{k} n_j}\left(\Delta g\right)$$

Proof

R1	Let		$x_k = \epsilon_{\mathrm{g},n_k}\left(\ldots \epsilon_{\mathrm{g},n_2}\left(\epsilon_{\mathrm{g},n_1}(\Delta g)\right)\ldots\right)$
R2	Let		$y_k = \epsilon_{\mathrm{g},\prod_{j=1}^{k} n_j}(\Delta g)$
R3			$\prod_{j=1}^{1} n_j = n_1$
R4	R3	$\Rightarrow$	$\epsilon_{\mathrm{g},n_1}(\Delta g) = \epsilon_{\mathrm{g},\prod_{j=1}^{1} n_j}(\Delta g)$
R5	R1, R2 & R4	$\Rightarrow$	$y_k = x_k$ when $k = 1$
R6	R1 & R2	$\Rightarrow$	$\left(y_k = x_k \Rightarrow x_{k+1} = \epsilon_{\mathrm{g},n_{k+1}}(y_k)\right)$
R7	501	$\Rightarrow$	$\epsilon_{\mathrm{g},n_{k+1}}(y_k) = \begin{bmatrix} n_{k+1} \times \Delta\,\mathrm{g_c}(y_k) - \mu_\mathrm{c} \times ((n_{k+1} \times \Delta\,\mathrm{m}(y_k)) \text{ div } \mu_\mathrm{m}), \\ (n_{k+1} \times \Delta\,\mathrm{m}(y_k)) \bmod \mu_\mathrm{m} \end{bmatrix}$
R8	R2 & 501	$\Rightarrow$	$y_{k+1} = \begin{bmatrix} \left(\prod_{j=1}^{k+1} n_j\right) \times \Delta\,\mathrm{g_c}(\Delta g) - \mu_\mathrm{c} \times \left(\left(\left(\prod_{j=1}^{k+1} n_j\right) \times \Delta\,\mathrm{m}(\Delta g)\right) \text{ div } \mu_\mathrm{m}\right), \\ \left(\left(\prod_{j=1}^{k+1} n_j\right) \times \Delta\,\mathrm{m}(\Delta g)\right) \bmod \mu_\mathrm{m} \end{bmatrix}$
R9	R2 & 501	$\Rightarrow$	$y_k = \begin{bmatrix} \left(\prod_{j=1}^{k} n_j\right) \times \Delta\,\mathrm{g_c}(\Delta g) - \mu_\mathrm{c} \times \left(\left(\left(\prod_{j=1}^{k} n_j\right) \times \Delta\,\mathrm{m}(\Delta g)\right) \text{ div } \mu_\mathrm{m}\right), \\ \left(\left(\prod_{j=1}^{k} n_j\right) \times \Delta\,\mathrm{m}(\Delta g)\right) \bmod \mu_\mathrm{m} \end{bmatrix}$
R10	R7, R9, 310 & 316	$\Rightarrow$	$\epsilon_{\mathrm{g},n_{k+1}}(y_k)$

$$= \begin{bmatrix} n_{k+1} \times \left(\left(\prod_{j=1}^{k} n_j\right) \times \Delta\,\mathrm{g_c}(\Delta g) - \mu_\mathrm{c} \times \left(\left(\left(\prod_{j=1}^{k} n_j\right) \times \Delta\,\mathrm{m}(\Delta g)\right) \text{ div } \mu_\mathrm{m}\right)\right) \\ -\mu_\mathrm{c} \times \left(\left(n_{k+1} \times \left(\left(\left(\prod_{j=1}^{k} n_j\right) \times \Delta\,\mathrm{m}(\Delta g)\right) \bmod \mu_\mathrm{m}\right)\right) \text{ div } \mu_\mathrm{m}\right), \\ \left(n_{k+1} \times \left(\left(\left(\prod_{j=1}^{k} n_j\right) \times \Delta\,\mathrm{m}(\Delta g)\right) \bmod \mu_\mathrm{m}\right)\right) \bmod \mu_\mathrm{m} \end{bmatrix}$$

$$= \begin{bmatrix} n_{k+1} \times \left(\prod_{j=1}^{k} n_j\right) \times \Delta\,\mathrm{g_c}(\Delta g) \\ -n_{k+1} \times \mu_\mathrm{c} \times \left(\left(\left(\prod_{j=1}^{k} n_j\right) \times \Delta\,\mathrm{m}(\Delta g)\right) \text{ div } \mu_\mathrm{m}\right) \\ -\mu_\mathrm{c} \times \left(\left(n_{k+1} \times \left(\left(\left(\prod_{j=1}^{k} n_j\right) \times \Delta\,\mathrm{m}(\Delta g)\right) \bmod \mu_\mathrm{m}\right)\right) \text{ div } \mu_\mathrm{m}\right), \\ \left(n_{k+1} \times \left(\left(\left(\prod_{j=1}^{k} n_j\right) \times \Delta\,\mathrm{m}(\Delta g)\right) \bmod \mu_\mathrm{m}\right)\right) \bmod \mu_\mathrm{m} \end{bmatrix}$$

(R10 cont.)

$$= \begin{bmatrix} \left(\prod_{j=1}^{k+1} n_j\right) \times \Delta\,\mathrm{g_c}\,(\Delta g) \\ -\mu_{\mathrm{c}} \times \begin{pmatrix} n_{k+1} \times \left(\left(\left(\prod_{j=1}^{k} n_j\right) \times \Delta\,\mathrm{m}\,(\Delta g)\right) \operatorname{div} \mu_{\mathrm{m}}\right) \\ + \left(n_{k+1} \times \left(\left(\left(\prod_{j=1}^{k} n_j\right) \times \Delta\,\mathrm{m}\,(\Delta g)\right) \bmod \mu_{\mathrm{m}}\right)\right) \operatorname{div} \mu_{\mathrm{m}} \end{pmatrix}, \\ \left(n_{k+1} \times \left(\left(\left(\prod_{j=1}^{k} n_j\right) \times \Delta\,\mathrm{m}\,(\Delta g)\right) \bmod \mu_{\mathrm{m}}\right)\right) \bmod \mu_{\mathrm{m}} \end{bmatrix}$$

R11 58 $\Rightarrow$

$$\begin{pmatrix} n_{k+1} \times \left(\left(\left(\prod_{j=1}^{k} n_j\right) \times \Delta\,\mathrm{m}\,(\Delta g)\right) \operatorname{div} \mu_{\mathrm{m}}\right) \\ + \left(n_{k+1} \times \left(\left(\left(\prod_{j=1}^{k} n_j\right) \times \Delta\,\mathrm{m}\,(\Delta g)\right) \bmod \mu_{\mathrm{m}}\right)\right) \operatorname{div} \mu_{\mathrm{m}} \end{pmatrix}$$

$$= \left(n_{k+1} \times \left(\prod_{j=1}^{k} n_j\right) \times \Delta\,\mathrm{m}\,(\Delta g)\right) \operatorname{div} \mu_{\mathrm{m}}$$

$$= \left(\left(\prod_{j=1}^{k+1} n_j\right) \times \Delta\,\mathrm{m}\,(\Delta g)\right) \operatorname{div} \mu_{\mathrm{m}}$$

R12 R11 & R10 $\Rightarrow$ $\epsilon_{\mathrm{g},n_{k+1}}(y_k)$

$$= \begin{bmatrix} \left(\prod_{j=1}^{k+1} n_j\right) \times \Delta\,\mathrm{g_c}\,(\Delta g) \\ -\mu_{\mathrm{c}} \times \left(\left(\left(\prod_{j=1}^{k+1} n_j\right) \times \Delta\,\mathrm{m}\,(\Delta g)\right) \operatorname{div} \mu_{\mathrm{m}}\right), \\ \left(n_{k+1} \times \left(\left(\left(\prod_{j=1}^{k} n_j\right) \times \Delta\,\mathrm{m}\,(\Delta g)\right) \bmod \mu_{\mathrm{m}}\right)\right) \bmod \mu_{\mathrm{m}} \end{bmatrix}$$

R13 45 $\Rightarrow$ $\left(n_{k+1} \times \left(\left(\left(\prod_{j=1}^{k} n_j\right) \times \Delta\,\mathrm{m}\,(\Delta g)\right) \bmod \mu_{\mathrm{m}}\right)\right) \bmod \mu_{\mathrm{m}}$

$$= \left(n_{k+1} \times \left(\prod_{j=1}^{k} n_j\right) \times \Delta\,\mathrm{m}\,(\Delta g)\right) \bmod \mu_{\mathrm{m}}$$

$$= \left(\left(\prod_{j=1}^{k+1} n_j\right) \times \Delta\,\mathrm{m}\,(\Delta g)\right) \bmod \mu_{\mathrm{m}}$$

R14 R13 & R12 $\Rightarrow$ $\epsilon_{\mathrm{g},n_{k+1}}(y_k)$

$$= \begin{bmatrix} \left(\prod_{j=1}^{k+1} n_j\right) \times \Delta\,\mathrm{g_c}\,(\Delta g) \\ -\mu_{\mathrm{c}} \times \left(\left(\left(\prod_{j=1}^{k+1} n_j\right) \times \Delta\,\mathrm{m}\,(\Delta g)\right) \operatorname{div} \mu_{\mathrm{m}}\right), \\ \left(\left(\prod_{j=1}^{k+1} n_j\right) \times \Delta\,\mathrm{m}\,(\Delta g)\right) \bmod \mu_{\mathrm{m}} \end{bmatrix}$$

R15 R14 & R8 $\Rightarrow$ $\epsilon_{\mathrm{g},n_{k+1}}(y_k) = y_{k+1}$

R16 R15 & R6 $\Rightarrow$ $(y_k = x_k \Rightarrow x_{k+1} = y_{k+1})$

R17 R16 & R5 $\Rightarrow$ $x_k = y_k$ for all integer k

R18 R17, R1 & R2 $\Rightarrow$ $\epsilon_{\mathrm{g},n_k}(\ldots\epsilon_{\mathrm{g},n_2}(\epsilon_{\mathrm{g},n_1}(\Delta g))\ldots) = \epsilon_{\mathrm{g},\prod_{j=1}^{k} n_j}(\Delta g)$

Theorem 504 *If*

$$\psi = [\mu_\mathrm{c}, \mu_\mathrm{m}, f_0, p_{\mathrm{c},0}]$$

is a pitch system, n is an integer and Δg is a genus interval in ψ then

$$\iota_\mathrm{g}(\epsilon_{\mathrm{g},n}(\Delta g)) = \epsilon_{\mathrm{g},-n}(\Delta g)$$

Proof

R1 502 $\Rightarrow$ $\iota_\mathrm{g}(\epsilon_{\mathrm{g},n}(\Delta g)) = \epsilon_{\mathrm{g},-1}(\epsilon_{\mathrm{g},n}(\Delta g))$

R2 503 $\Rightarrow$ $\epsilon_{\mathrm{g},-1}(\epsilon_{\mathrm{g},n}(\Delta g)) = \epsilon_{\mathrm{g},(-1\times n)}(\Delta g) = \epsilon_{\mathrm{g},-n}(\Delta g)$

R3 R1 & R2 $\Rightarrow$ $\iota_\mathrm{g}(\epsilon_{\mathrm{g},n}(\Delta g)) = \epsilon_{\mathrm{g},-n}(\Delta g)$

Theorem 505 *If*

$$\psi = [\mu_\mathrm{c}, \mu_\mathrm{m}, f_0, p_{\mathrm{c},0}]$$

is a pitch system, n is an integer and Δg is a genus interval in ψ then:

$$\Delta\,\mathrm{c}(\epsilon_{\mathrm{g},n}(\Delta g)) = \epsilon_{\mathrm{c},n}(\Delta\,\mathrm{c}(\Delta g))$$

Proof

R1 501 $\Rightarrow$ $\epsilon_{\mathrm{g},n}(\Delta g) = \begin{bmatrix} n\times\Delta\,\mathrm{g_c}(\Delta g) - \mu_\mathrm{c}\times((n\times\Delta\,\mathrm{m}(\Delta g)) \operatorname{div} \mu_\mathrm{m}), \\ (n\times\Delta\,\mathrm{m}(\Delta g)) \bmod \mu_\mathrm{m} \end{bmatrix}$

R2 R1, 313 & 310 $\Rightarrow$ $\Delta\,\mathrm{c}(\epsilon_{\mathrm{g},n}(\Delta g)) = (n\times\Delta\,\mathrm{g_c}(\Delta g) - \mu_\mathrm{c}\times((n\times\Delta\,\mathrm{m}(\Delta g)) \operatorname{div} \mu_\mathrm{m})) \bmod \mu_\mathrm{c}$

R3 313 $\Rightarrow$ $\epsilon_{\mathrm{c},n}(\Delta\,\mathrm{c}(\Delta g)) = \epsilon_{\mathrm{c},n}(\Delta\,\mathrm{g_c}(\Delta g) \bmod \mu_\mathrm{c})$

R4 R3 & 454 $\Rightarrow$ $\epsilon_{\mathrm{c},n}(\Delta\,\mathrm{c}(\Delta g)) = (n\times(\Delta\,\mathrm{g_c}(\Delta g) \bmod \mu_\mathrm{c})) \bmod \mu_\mathrm{c}$

R5 R4 & 45 $\Rightarrow$ $\epsilon_{\mathrm{c},n}(\Delta\,\mathrm{c}(\Delta g)) = (n\times\Delta\,\mathrm{g_c}(\Delta g)) \bmod \mu_\mathrm{c}$

R6 R2 & 37 $\Rightarrow$ $\Delta\,\mathrm{c}(\epsilon_{\mathrm{g},n}(\Delta g)) = (n\times\Delta\,\mathrm{g_c}(\Delta g)) \bmod \mu_\mathrm{c}$

R7 R5 & R6 $\Rightarrow$ $\Delta\,\mathrm{c}(\epsilon_{\mathrm{g},n}(\Delta g)) = \epsilon_{\mathrm{c},n}(\Delta\,\mathrm{c}(\Delta g))$

Theorem 506 *If*

$$\psi = [\mu_\mathrm{c}, \mu_\mathrm{m}, f_0, p_{\mathrm{c},0}]$$

is a pitch system, n is an integer and Δg is a genus interval in ψ then:

$$\Delta\,\mathrm{m}(\epsilon_{\mathrm{g},n}(\Delta g)) = \epsilon_{\mathrm{m},n}(\Delta\,\mathrm{m}(\Delta g))$$

Proof

R1	501	$\Rightarrow$	$\epsilon_{\mathrm{g},n}(\Delta g) = \begin{bmatrix} n \times \Delta\,\mathrm{g_c}(\Delta g) - \mu_\mathrm{c} \times ((n \times \Delta\,\mathrm{m}(\Delta g)) \operatorname{div} \mu_\mathrm{m}), \\ (n \times \Delta\,\mathrm{m}(\Delta g)) \bmod \mu_\mathrm{m} \end{bmatrix}$
R2	R1 & 316	$\Rightarrow$	$\Delta\,\mathrm{m}(\epsilon_{\mathrm{g},n}(\Delta g)) = (n \times \Delta\,\mathrm{m}(\Delta g)) \bmod \mu_\mathrm{m}$
R3	468	$\Rightarrow$	$\epsilon_{\mathrm{m},n}(\Delta\,\mathrm{m}(\Delta g)) = (n \times \Delta\,\mathrm{m}(\Delta g)) \bmod \mu_\mathrm{m}$
R4	R2 & R3	$\Rightarrow$	$\Delta\,\mathrm{m}(\epsilon_{\mathrm{g},n}(\Delta g)) = \epsilon_{\mathrm{m},n}(\Delta\,\mathrm{m}(\Delta g))$

Theorem 507 *If*

$$\psi = [\mu_\mathrm{c}, \mu_\mathrm{m}, f_0, p_{\mathrm{c},0}]$$

is a pitch system, n is an integer and Δg is a genus interval in ψ then:

$$\Delta\,\mathrm{q}(\epsilon_{\mathrm{g},n}(\Delta g)) = \epsilon_{\mathrm{q},n}(\Delta\,\mathrm{q}(\Delta g))$$

Proof

R1	501	$\Rightarrow$	$\epsilon_{\mathrm{g},n}(\Delta g) = \begin{bmatrix} n \times \Delta\,\mathrm{g_c}(\Delta g) - \mu_\mathrm{c} \times ((n \times \Delta\,\mathrm{m}(\Delta g)) \operatorname{div} \mu_\mathrm{m}), \\ (n \times \Delta\,\mathrm{m}(\Delta g)) \bmod \mu_\mathrm{m} \end{bmatrix}$
R2	R1 & 320	$\Rightarrow$	$\Delta\,\mathrm{q}(\epsilon_{\mathrm{g},n}(\Delta g)) = [\Delta\,\mathrm{c}(\epsilon_{\mathrm{g},n}(\Delta g)), \Delta\,\mathrm{m}(\epsilon_{\mathrm{g},n}(\Delta g))]$
R3	R2 & 505	$\Rightarrow$	$\Delta\,\mathrm{q}(\epsilon_{\mathrm{g},n}(\Delta g)) = [\epsilon_{\mathrm{c},n}(\Delta\,\mathrm{c}(\Delta g)), \Delta\,\mathrm{m}(\epsilon_{\mathrm{g},n}(\Delta g))]$
R4	R3 & 506	$\Rightarrow$	$\Delta\,\mathrm{q}(\epsilon_{\mathrm{g},n}(\Delta g)) = [\epsilon_{\mathrm{c},n}(\Delta\,\mathrm{c}(\Delta g)), \epsilon_{\mathrm{m},n}(\Delta\,\mathrm{m}(\Delta g))]$
R5	320	$\Rightarrow$	$\Delta\,\mathrm{q}(\Delta g) = [\Delta\,\mathrm{c}(\Delta g), \Delta\,\mathrm{m}(\Delta g)]$
R6	R5 & 300	$\Rightarrow$	$\Delta\,\mathrm{c}(\Delta\,\mathrm{q}(\Delta g)) = \Delta\,\mathrm{c}(\Delta g)$
R7	R5 & 303	$\Rightarrow$	$\Delta\,\mathrm{m}(\Delta\,\mathrm{q}(\Delta g)) = \Delta\,\mathrm{m}(\Delta g)$
R8	R4, R6 & R7	$\Rightarrow$	$\Delta\,\mathrm{q}(\epsilon_{\mathrm{g},n}(\Delta g)) = [\epsilon_{\mathrm{c},n}(\Delta\,\mathrm{c}(\Delta\,\mathrm{q}(\Delta g))), \epsilon_{\mathrm{m},n}(\Delta\,\mathrm{m}(\Delta\,\mathrm{q}(\Delta g)))]$
R9	R8 & 482	$\Rightarrow$	$\Delta\,\mathrm{q}(\epsilon_{\mathrm{g},n}(\Delta g)) = \epsilon_{\mathrm{q},n}(\Delta\,\mathrm{q}(\Delta g))$

Theorem 508 *If*

$$\psi = [\mu_\mathrm{c}, \mu_\mathrm{m}, f_0, p_{\mathrm{c},0}]$$

is a pitch system, $n_1, n_2, \ldots n_k$ is a collection of integers and Δg is a genus interval in ψ then

$$\sigma_\mathrm{g}(\epsilon_{\mathrm{g},n_1}(\Delta g), \epsilon_{\mathrm{g},n_2}(\Delta g), \ldots, \epsilon_{\mathrm{g},n_k}(\Delta g)) = \epsilon_{\mathrm{g},\sum_{j=1}^{k} n_j}(\Delta g)$$

Proof

R1 Let $x_k = \sigma_{\mathrm{g}}\left(\epsilon_{\mathrm{g},n_1}(\Delta g), \epsilon_{\mathrm{g},n_2}(\Delta g), \ldots, \epsilon_{\mathrm{g},n_k}(\Delta g)\right)$

R2 Let $y_k = \epsilon_{\mathrm{g},\sum_{j=1}^{k} n_j}(\Delta g)$

R3 R1 & 491 $\Rightarrow$ $$x_k = \begin{bmatrix} \sum_{j=1}^{k} \Delta\,\mathrm{g_c}\left(\epsilon_{\mathrm{g},n_j}(\Delta g)\right) - \mu_{\mathrm{c}} \times \left(\left(\sum_{j=1}^{k} \Delta\,\mathrm{m}\left(\epsilon_{\mathrm{g},n_j}(\Delta g)\right)\right) \mathrm{div}\ \mu_{\mathrm{m}}\right), \\ \left(\sum_{j=1}^{k} \Delta\,\mathrm{m}\left(\epsilon_{\mathrm{g},n_j}(\Delta g)\right)\right) \mathrm{mod}\ \mu_{\mathrm{m}} \end{bmatrix}$$

R4 R2 & 501 $\Rightarrow$ $$y_k = \begin{bmatrix} \sum_{j=1}^{k} n_j \times \Delta\,\mathrm{g_c}(\Delta g) - \mu_{\mathrm{c}} \times \left(\left(\left(\sum_{j=1}^{k} n_j\right) \times \Delta\,\mathrm{m}(\Delta g)\right) \mathrm{div}\ \mu_{\mathrm{m}}\right), \\ \left(\left(\sum_{j=1}^{k} n_j\right) \times \Delta\,\mathrm{m}(\Delta g)\right) \mathrm{mod}\ \mu_{\mathrm{m}} \end{bmatrix}$$

R5 501 $\Rightarrow$ $$\epsilon_{\mathrm{g},n_j}(\Delta g) = \begin{bmatrix} n_j \times \Delta\,\mathrm{g_c}(\Delta g) - \mu_{\mathrm{c}} \times \left((n_j \times \Delta\,\mathrm{m}(\Delta g))\ \mathrm{div}\ \mu_{\mathrm{m}}\right), \\ (n_j \times \Delta\,\mathrm{m}(\Delta g))\ \mathrm{mod}\ \mu_{\mathrm{m}} \end{bmatrix}$$

R6 R5 & 310 $\Rightarrow$ $\Delta\,\mathrm{g_c}\left(\epsilon_{\mathrm{g},n_j}(\Delta g)\right) = n_j \times \Delta\,\mathrm{g_c}(\Delta g) - \mu_{\mathrm{c}} \times \left((n_j \times \Delta\,\mathrm{m}(\Delta g))\ \mathrm{div}\ \mu_{\mathrm{m}}\right)$

R7 R5 & 316 $\Rightarrow$ $\Delta\,\mathrm{m}\left(\epsilon_{\mathrm{g},n_j}(\Delta g)\right) = (n_j \times \Delta\,\mathrm{m}(\Delta g))\ \mathrm{mod}\ \mu_{\mathrm{m}}$

R8 R6 $\Rightarrow$ $$\sum_{j=1}^{k} \Delta\,\mathrm{g_c}\left(\epsilon_{\mathrm{g},n_j}(\Delta g)\right) = \sum_{j=1}^{k} \left(n_j \times \Delta\,\mathrm{g_c}(\Delta g)\right) - \mu_{\mathrm{c}} \times \sum_{j=1}^{k} \left((n_j \times \Delta\,\mathrm{m}(\Delta g))\ \mathrm{div}\ \mu_{\mathrm{m}}\right)$$
$$= \left(\sum_{j=1}^{k} n_j\right) \times \Delta\,\mathrm{g_c}(\Delta g) - \mu_{\mathrm{c}} \times \sum_{j=1}^{k} \left((n_j \times \Delta\,\mathrm{m}(\Delta g))\ \mathrm{div}\ \mu_{\mathrm{m}}\right)$$

R9 R7 $\Rightarrow$ $\sum_{j=1}^{k} \Delta\,\mathrm{m}\left(\epsilon_{\mathrm{g},n_j}(\Delta g)\right) = \sum_{j=1}^{k} \left((n_j \times \Delta\,\mathrm{m}(\Delta g))\ \mathrm{mod}\ \mu_{\mathrm{m}}\right)$

R10 R3, R8 & R9 $\Rightarrow$ $$x_k = \begin{bmatrix} \left(\sum_{j=1}^{k} n_j\right) \times \Delta\,\mathrm{g_c}(\Delta g) - \mu_{\mathrm{c}} \times \sum_{j=1}^{k} \left((n_j \times \Delta\,\mathrm{m}(\Delta g))\ \mathrm{div}\ \mu_{\mathrm{m}}\right) \\ -\mu_{\mathrm{c}} \times \left(\left(\sum_{j=1}^{k} \left((n_j \times \Delta\,\mathrm{m}(\Delta g))\ \mathrm{mod}\ \mu_{\mathrm{m}}\right)\right) \mathrm{div}\ \mu_{\mathrm{m}}\right), \\ \left(\sum_{j=1}^{k} \left((n_j \times \Delta\,\mathrm{m}(\Delta g))\ \mathrm{mod}\ \mu_{\mathrm{m}}\right)\right) \mathrm{mod}\ \mu_{\mathrm{m}} \end{bmatrix}$$

$$= \begin{bmatrix} \left(\sum_{j=1}^{k} n_j\right) \times \Delta\,\mathrm{g_c}(\Delta g) \\ -\mu_{\mathrm{c}} \times \begin{pmatrix} \sum_{j=1}^{k} \left((n_j \times \Delta\,\mathrm{m}(\Delta g))\ \mathrm{div}\ \mu_{\mathrm{m}}\right) \\ + \left(\sum_{j=1}^{k} \left((n_j \times \Delta\,\mathrm{m}(\Delta g))\ \mathrm{mod}\ \mu_{\mathrm{m}}\right)\right) \mathrm{div}\ \mu_{\mathrm{m}} \end{pmatrix}, \\ \left(\sum_{j=1}^{k} \left((n_j \times \Delta\,\mathrm{m}(\Delta g))\ \mathrm{mod}\ \mu_{\mathrm{m}}\right)\right) \mathrm{mod}\ \mu_{\mathrm{m}} \end{bmatrix}$$

R11	54	$\Rightarrow$	$\sum_{j=1}^{k}((n_j \times \Delta\,\mathrm{m}(\Delta g)) \operatorname{div} \mu_{\mathrm{m}}) + \left(\sum_{j=1}^{k}((n_j \times \Delta\,\mathrm{m}(\Delta g)) \bmod \mu_{\mathrm{m}})\right) \operatorname{div} \mu_{\mathrm{m}}$
			$= \left(\Delta\,\mathrm{m}(\Delta g) \times \sum_{j=1}^{k} n_j\right) \operatorname{div} \mu_{\mathrm{m}}$
R12	R10 & R11	$\Rightarrow$	$x_k = \begin{bmatrix} \left(\sum_{j=1}^{k} n_j\right) \times \Delta\,\mathrm{g_c}(\Delta g) \\ -\mu_{\mathrm{c}} \times \left(\left(\Delta\,\mathrm{m}(\Delta g) \times \sum_{j=1}^{k} n_j\right) \operatorname{div} \mu_{\mathrm{m}}\right), \\ \left(\sum_{j=1}^{k}((n_j \times \Delta\,\mathrm{m}(\Delta g)) \bmod \mu_{\mathrm{m}})\right) \bmod \mu_{\mathrm{m}} \end{bmatrix}$
R13	39	$\Rightarrow$	$\left(\sum_{j=1}^{k}((n_j \times \Delta\,\mathrm{m}(\Delta g)) \bmod \mu_{\mathrm{m}})\right) \bmod \mu_{\mathrm{m}} = \left(\left(\sum_{j=1}^{k} n_j\right) \times \Delta\,\mathrm{m}(\Delta g)\right) \bmod \mu_{\mathrm{m}}$
R14	R12 & R13	$\Rightarrow$	$x_k = \begin{bmatrix} \left(\sum_{j=1}^{k} n_j\right) \times \Delta\,\mathrm{g_c}(\Delta g) \\ -\mu_{\mathrm{c}} \times \left(\left(\Delta\,\mathrm{m}(\Delta g) \times \sum_{j=1}^{k} n_j\right) \operatorname{div} \mu_{\mathrm{m}}\right), \\ \left(\left(\sum_{j=1}^{k} n_j\right) \times \Delta\,\mathrm{m}(\Delta g)\right) \bmod \mu_{\mathrm{m}} \end{bmatrix}$
R15	R4 & R14	$\Rightarrow$	$x_k = y_k$
R16	R1, R2 & R15	$\Rightarrow$	$\sigma_{\mathrm{g}}(\epsilon_{\mathrm{g},n_1}(\Delta g), \epsilon_{\mathrm{g},n_2}(\Delta g), \ldots, \epsilon_{\mathrm{g},n_k}(\Delta g)) = \epsilon_{\mathrm{g},\sum_{j=1}^{k} n_j}(\Delta g)$

Exponentiation of the genus tranposition function

Definition 509 (Definition of $\tau_{\mathrm{g},n}(g, \Delta g)$) *If ψ is a pitch system and g is a genus in ψ and Δg is a genus interval in ψ then*

$$\tau_{\mathrm{g},n}(g, \Delta g) = \tau_{\mathrm{g}}(g, \epsilon_{\mathrm{g},n}(\Delta g))$$

Theorem 510 *If*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

is a pitch system, $n_1, n_2, \ldots, n_k$ is a collection of integers, g is a genus in ψ and Δg is a genus interval in ψ then

$$\tau_{\mathrm{g},n_k}(\ldots \tau_{\mathrm{g},n_2}(\tau_{\mathrm{g},n_1}(g, \Delta g), \Delta g) \ldots, \Delta g) = \tau_{\mathrm{g},\sum_{j=1}^{k} n_j}(g, \Delta g)$$

Proof

R1	Let		$x_k = \tau_{\mathrm{g},n_k}\left(\ldots\tau_{\mathrm{g},n_2}\left(\tau_{\mathrm{g},n_1}\left(g,\Delta g\right),\Delta g\right)\ldots,\Delta g\right)$
R2	R1 & 509	$\Rightarrow$	$x_k = \tau_{\mathrm{g}}\left(\ldots\tau_{\mathrm{g}}\left(\tau_{\mathrm{g}}\left(g,\epsilon_{\mathrm{g},n_1}\left(\Delta g\right)\right),\epsilon_{\mathrm{g},n_2}\left(\Delta g\right)\right)\ldots,\epsilon_{\mathrm{g},n_k}\left(\Delta g\right)\right)$
R3	R2 & 493	$\Rightarrow$	$x_k = \tau_{\mathrm{g}}\left(g,\sigma_{\mathrm{g}}\left(\epsilon_{\mathrm{g},n_1}\left(\Delta g\right),\epsilon_{\mathrm{g},n_2}\left(\Delta g\right),\ldots,\epsilon_{\mathrm{g},n_k}\left(\Delta g\right)\right)\right)$
R4	R3 & 508	$\Rightarrow$	$x_k = \tau_{\mathrm{g}}\left(g,\epsilon_{\mathrm{g},\sum_{j=1}^{k} n_j}\left(\Delta g\right)\right)$
R5	R1, R4 & 509	$\Rightarrow$	$\tau_{\mathrm{g},n_k}\left(\ldots\tau_{\mathrm{g},n_2}\left(\tau_{\mathrm{g},n_1}\left(g,\Delta g\right),\Delta g\right)\ldots,\Delta g\right) = \tau_{\mathrm{g},\sum_{j=1}^{k} n_j}\left(g,\Delta g\right)$

4.6.5 Summation, inversion and exponentiation of chromatic pitch intervals

Summation of chromatic pitch intervals

Definition 511 *If*

$$\psi = [\mu_{\mathrm{c}},\mu_{\mathrm{m}},f_0,p_{\mathrm{c},0}]$$

is a pitch system and

$$\Delta p_{\mathrm{c},1},\Delta p_{\mathrm{c},2},\ldots\Delta p_{\mathrm{c},n}$$

is a collection of chromatic pitch intervals in ψ then

$$\sigma_{\mathrm{p_c}}\left(\Delta p_{\mathrm{c},1},\Delta p_{\mathrm{c},2},\ldots\Delta p_{\mathrm{c},n}\right) = \sum_{k=1}^{n}\Delta p_{\mathrm{c},k}$$

Theorem 512 *If ψ is a pitch system and*

$$\Delta p_{\mathrm{c},1},\Delta p_{\mathrm{c},2},\ldots\Delta p_{\mathrm{c},n}$$

is a collection of chromatic pitch intervals in ψ and p_{c} is a chromatic pitch in ψ then

$$\tau_{\mathrm{p_c}}\left(p_{\mathrm{c}},\sigma_{\mathrm{p_c}}\left(\Delta p_{\mathrm{c},1},\Delta p_{\mathrm{c},2},\ldots\Delta p_{\mathrm{c},n}\right)\right) = \tau_{\mathrm{p_c}}\left(\ldots\tau_{\mathrm{p_c}}\left(\tau_{\mathrm{p_c}}\left(p_{\mathrm{c}},\Delta p_{\mathrm{c},1}\right),\Delta p_{\mathrm{c},2}\right)\ldots,\Delta p_{\mathrm{c},n}\right)$$

Proof

R1	Let		$x_n = \tau_{\mathrm{p_c}}\left(\ldots \tau_{\mathrm{p_c}}\left(\tau_{\mathrm{p_c}}\left(p_{\mathrm{c}}, \Delta p_{\mathrm{c},1}\right), \Delta p_{\mathrm{c},2}\right) \ldots, \Delta p_{\mathrm{c},n}\right)$
R2	Let		$y_n = \tau_{\mathrm{p_c}}\left(p_{\mathrm{c}}, \sigma_{\mathrm{p_c}}\left(\Delta p_{\mathrm{c},1}, \Delta p_{\mathrm{c},2}, \ldots \Delta p_{\mathrm{c},n}\right)\right)$
R3	R1 & 427	$\Rightarrow$	$x_1 = \tau_{\mathrm{p_c}}\left(p_{\mathrm{c}}, \Delta p_{\mathrm{c},1}\right) = p_{\mathrm{c}} + \Delta p_{\mathrm{c},1}$
R4	R2 & 427	$\Rightarrow$	$y_1 = p_{\mathrm{c}} + \sigma_{\mathrm{p_c}}\left(\Delta p_{\mathrm{c},1}\right)$
R5	R4 & 511	$\Rightarrow$	$y_1 = p_{\mathrm{c}} + \Delta p_{\mathrm{c},1}$
R6	R3 & R5	$\Rightarrow$	$x_1 = y_1$
R7	R1	$\Rightarrow$	$\left(x_k = y_k \Rightarrow x_{k+1} = \tau_{\mathrm{p_c}}\left(y_k, \Delta p_{\mathrm{c},k+1}\right)\right)$
R8	R2 & 427	$\Rightarrow$	$y_n = p_{\mathrm{c}} + \sigma_{\mathrm{p_c}}\left(\Delta p_{\mathrm{c},1}, \Delta p_{\mathrm{c},2}, \ldots \Delta p_{\mathrm{c},n}\right)$
R9	R8 & 511	$\Rightarrow$	$y_n = p_{\mathrm{c}} + \sum_{k=1}^{n} \Delta p_{\mathrm{c},k}$
R10	427	$\Rightarrow$	$\tau_{\mathrm{p_c}}\left(y_k, \Delta p_{\mathrm{c},k+1}\right) = y_k + \Delta p_{\mathrm{c},k+1}$
R11	R9 & R10	$\Rightarrow$	$\tau_{\mathrm{p_c}}\left(y_k, \Delta p_{\mathrm{c},k+1}\right) = p_{\mathrm{c}} + \sum_{j=1}^{k} \Delta p_{\mathrm{c},j} + \Delta p_{\mathrm{c},k+1}$
			$= p_{\mathrm{c}} + \sum_{j=1}^{k+1} \Delta p_{\mathrm{c},j}$
R12	R11 & R9	$\Rightarrow$	$\tau_{\mathrm{p_c}}\left(y_k, \Delta p_{\mathrm{c},k+1}\right) = y_{k+1}$
R13	R12 & R7	$\Rightarrow$	$\left(x_k = y_k \Rightarrow x_{k+1} = y_{k+1}\right)$
R14	R6 & R13	$\Rightarrow$	$x_k = y_k$ for all positive integers k
R15	R1, R2 & R14	$\Rightarrow$	$\tau_{\mathrm{p_c}}\left(p_{\mathrm{c}}, \sigma_{\mathrm{p_c}}\left(\Delta p_{\mathrm{c},1}, \Delta p_{\mathrm{c},2}, \ldots \Delta p_{\mathrm{c},n}\right)\right) = \tau_{\mathrm{p_c}}\left(\ldots \tau_{\mathrm{p_c}}\left(\tau_{\mathrm{p_c}}\left(p_{\mathrm{c}}, \Delta p_{\mathrm{c},1}\right), \Delta p_{\mathrm{c},2}\right) \ldots, \Delta p_{\mathrm{c},n}\right)$

Inversion of chromatic pitch intervals

Definition 513 (Definition of $\iota_{\mathrm{p_c}}\left(\Delta p_{\mathrm{c}}\right)$) *If ψ is a pitch system and Δp_{c} is a chromatic pitch interval in ψ and p_{c} is a chromatic pitch in ψ then $\iota_{\mathrm{p_c}}\left(\Delta p_{\mathrm{c}}\right)$ is the chromatic pitch interval that satisfies the following equation*

$$\tau_{\mathrm{p_c}}\left(\tau_{\mathrm{p_c}}\left(p_{\mathrm{c}}, \Delta p_{\mathrm{c}}\right), \iota_{\mathrm{p_c}}\left(\Delta p_{\mathrm{c}}\right)\right) = p_{\mathrm{c}}$$

Definition 514 (Inversional equivalence of chromatic pitch intervals) *If ψ is a pitch system and $\Delta p_{\mathrm{c},1}$ and $\Delta p_{\mathrm{c},2}$ are chromatic pitch intervals in ψ then $\Delta p_{\mathrm{c},1}$ and $\Delta p_{\mathrm{c},2}$ are* inversionally equivalent *if and only if*

$$\left(\iota_{\mathrm{p_c}}\left(\Delta p_{\mathrm{c},1}\right) = \Delta p_{\mathrm{c},2}\right) \vee \left(\Delta p_{\mathrm{c},1} = \Delta p_{\mathrm{c},2}\right)$$

The fact that two chromatic pitch intervals are inversionally equivalent is denoted as follows:

$$\Delta p_{\mathrm{c},1} \equiv_{\iota} \Delta p_{\mathrm{c},2}$$

Theorem 515 *If*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

is a pitch system and Δp_{c} *is a chromatic pitch interval in* ψ *then*

$$\iota_{p_{\mathrm{c}}}(\Delta p_{\mathrm{c}}) = -\Delta p_{\mathrm{c}}$$

Proof

R1	513	$\Rightarrow$	$\tau_{p_{\mathrm{c}}}(\tau_{p_{\mathrm{c}}}(p_{\mathrm{c}}, \Delta p_{\mathrm{c}}), \iota_{p_{\mathrm{c}}}(\Delta p_{\mathrm{c}})) = p_{\mathrm{c}}$
R2	R1 & 427	$\Rightarrow$	$\tau_{p_{\mathrm{c}}}(p_{\mathrm{c}} + \Delta p_{\mathrm{c}}, \iota_{p_{\mathrm{c}}}(\Delta p_{\mathrm{c}})) = p_{\mathrm{c}}$
		$\Rightarrow$	$p_{\mathrm{c}} + \Delta p_{\mathrm{c}} + \iota_{p_{\mathrm{c}}}(\Delta p_{\mathrm{c}}) = p_{\mathrm{c}}$
		$\Rightarrow$	$\Delta p_{\mathrm{c}} + \iota_{p_{\mathrm{c}}}(\Delta p_{\mathrm{c}}) = 0$
		$\Rightarrow$	$\iota_{p_{\mathrm{c}}}(\Delta p_{\mathrm{c}}) = -\Delta p_{\mathrm{c}}$

Theorem 516 *If* ψ *is a pitch system and* Δp_{c}*,* $\Delta p_{\mathrm{c},1}$ *and* $\Delta p_{\mathrm{c},2}$ *are chromatic pitch intervals in* ψ *then*

$$(\Delta p_{\mathrm{c},1} = \iota_{p_{\mathrm{c}}}(\Delta p_{\mathrm{c}})) \wedge (\Delta p_{\mathrm{c},2} = \iota_{p_{\mathrm{c}}}(\Delta p_{\mathrm{c}})) \Rightarrow (\Delta p_{\mathrm{c},1} = \Delta p_{\mathrm{c},2})$$

Proof

R1	Let		$(\Delta p_{\mathrm{c},1} = \iota_{p_{\mathrm{c}}}(\Delta p_{\mathrm{c}})) \wedge (\Delta p_{\mathrm{c},2} = \iota_{p_{\mathrm{c}}}(\Delta p_{\mathrm{c}}))$
R2	R1 & 515	$\Rightarrow$	$\Delta p_{\mathrm{c},1} = -\Delta p_{\mathrm{c}}$
R3	R1 & 515	$\Rightarrow$	$\Delta p_{\mathrm{c},2} = -\Delta p_{\mathrm{c}}$
R4	R2 & R3	$\Rightarrow$	$\Delta p_{\mathrm{c},1} = \Delta p_{\mathrm{c},2}$
R5	R1 to R4	$\Rightarrow$	$(\Delta p_{\mathrm{c},1} = \iota_{p_{\mathrm{c}}}(\Delta p_{\mathrm{c}})) \wedge (\Delta p_{\mathrm{c},2} = \iota_{p_{\mathrm{c}}}(\Delta p_{\mathrm{c}})) \Rightarrow (\Delta p_{\mathrm{c},1} = \Delta p_{\mathrm{c},2})$

Exponentiation of chromatic pitch intervals

Definition 517 (Definition of $\epsilon_{p_{\mathrm{c}},n}(\Delta p_{\mathrm{c}})$) *Given that:*

1. ψ *is a pitch system;*
2. p_{c} *is a chromatic pitch in* ψ*;*
3. Δp_{c} *is a chromatic pitch interval in* ψ*;*
4. n *is an integer;*

5. *k is an integer and* $1 \leq k \leq \mathrm{abs}\,(n)$*;*

6. $\Delta p_{\mathrm{c},1,k} = \Delta p_{\mathrm{c}}$ *for all* k*; and*

7. $\Delta p_{\mathrm{c},2,k} = \iota_{\mathrm{p_c}}\,(\Delta p_{\mathrm{c}})$ *for all* k*;*

then $\epsilon_{\mathrm{p_c},n}\,(\Delta p_{\mathrm{c}})$ *returns a chromatic pitch interval that satisfies the following equation:*

$$\tau_{\mathrm{p_c}}\,(p_{\mathrm{c}}, \epsilon_{\mathrm{p_c},n}\,(\Delta p_{\mathrm{c}})) = \begin{cases} \tau_{\mathrm{p_c}}\,(p_{\mathrm{c}}, \sigma_{\mathrm{p_c}}\,(\Delta p_{\mathrm{c},1,1}, \Delta p_{\mathrm{c},1,2}, \ldots \Delta p_{\mathrm{c},1,n})) & \text{if} \quad n > 0 \\ p_{\mathrm{c}} & \text{if} \quad n = 0 \\ \tau_{\mathrm{p_c}}\,(p_{\mathrm{c}}, \sigma_{\mathrm{p_c}}\,(\Delta p_{\mathrm{c},2,1}, \Delta p_{\mathrm{c},2,2}, \ldots \Delta p_{\mathrm{c},2,-n})) & \text{if} \quad n < 0 \end{cases}$$

Theorem 518 (Formula for $\epsilon_{\mathrm{p_c},n}\,(\Delta p_{\mathrm{c}})$) *If*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

is a pitch system and Δp_{c} *is a chromatic pitch interval in* ψ *and* n *is an integer then*

$$\epsilon_{\mathrm{p_c},n}\,(\Delta p_{\mathrm{c}}) = n \times \Delta p_{\mathrm{c}}$$

Proof

R1	Let		n be any integer
R2	Let		k be an integer such that $1 \leq k \leq \mathrm{abs}\,(n)$
R3	Let		$\Delta p_{\mathrm{c},1,k} = \Delta p_{\mathrm{c}}$ for all k
R4	Let		$\Delta p_{\mathrm{c},2,k} = \iota_{\mathrm{p_c}}\,(\Delta p_{\mathrm{c}})$ for all k
R5	Let		n_1 be any integer greater than zero
R6	R3, R5 & 511	$\Rightarrow$	$\tau_{\mathrm{p_c}}\,(p_{\mathrm{c}}, \sigma_{\mathrm{p_c}}\,(\Delta p_{\mathrm{c},1,1}, \Delta p_{\mathrm{c},1,2}, \ldots \Delta p_{\mathrm{c},1,n_1}))$
			$= \tau_{\mathrm{p_c}}\left(p_{\mathrm{c}}, \sum_{j=1}^{n_1} \Delta p_{\mathrm{c},1,j}\right)$
			$= \tau_{\mathrm{p_c}}\,(p_{\mathrm{c}}, n_1 \times \Delta p_{\mathrm{c}})$
R7	427	$\Rightarrow$	$\tau_{\mathrm{p_c}}\,(p_{\mathrm{c}}, 0 \times \Delta p_{\mathrm{c}}) = p_{\mathrm{c}} + 0 \times \Delta p_{\mathrm{c}} = p_{\mathrm{c}}$
R8	Let		n_2 be any integer less than zero
R9	R4, R8 & 511	$\Rightarrow$	$\tau_{\mathrm{p_c}}\,(p_{\mathrm{c}}, \sigma_{\mathrm{p_c}}\,(\Delta p_{\mathrm{c},2,1}, \Delta p_{\mathrm{c},2,2}, \ldots \Delta p_{\mathrm{c},2,-n_2}))$
			$= \tau_{\mathrm{p_c}}\left(p_{\mathrm{c}}, \sum_{j=1}^{-n_2} \Delta p_{\mathrm{c},2,j}\right)$
			$= \tau_{\mathrm{p_c}}\,(p_{\mathrm{c}}, -n_2 \times \iota_{\mathrm{p_c}}\,(\Delta p_{\mathrm{c}}))$
R10	R9 & 515	$\Rightarrow$	$\tau_{\mathrm{p_c}}\,(p_{\mathrm{c}}, \sigma_{\mathrm{p_c}}\,(\Delta p_{\mathrm{c},2,1}, \Delta p_{\mathrm{c},2,2}, \ldots \Delta p_{\mathrm{c},2,-n_2}))$
			$= \tau_{\mathrm{p_c}}\,(p_{\mathrm{c}}, -n_2 \times (-\Delta p_{\mathrm{c}}))$
			$= \tau_{\mathrm{p_c}}\,(p_{\mathrm{c}}, n_2 \times \Delta p_{\mathrm{c}})$
R11	R1, R5 & R6	$\Rightarrow$	$\tau_{\mathrm{p_c}}\,(p_{\mathrm{c}}, \sigma_{\mathrm{p_c}}\,(\Delta p_{\mathrm{c},1,1}, \Delta p_{\mathrm{c},1,2}, \ldots \Delta p_{\mathrm{c},1,n})) = \tau_{\mathrm{p_c}}\,(p_{\mathrm{c}}, n \times \Delta p_{\mathrm{c}})$ when $n > 0$
R12	R1 & R7	$\Rightarrow$	$p_{\mathrm{c}} = \tau_{\mathrm{p_c}}\,(p_{\mathrm{c}}, n \times \Delta p_{\mathrm{c}})$ when $n = 0$
R13	R1, R8 & R10	$\Rightarrow$	$\tau_{\mathrm{p_c}}\,(p_{\mathrm{c}}, \sigma_{\mathrm{p_c}}\,(\Delta p_{\mathrm{c},2,1}, \Delta p_{\mathrm{c},2,2}, \ldots \Delta p_{\mathrm{c},2,-n})) = \tau_{\mathrm{p_c}}\,(p_{\mathrm{c}}, n \times \Delta p_{\mathrm{c}})$ when $n < 0$
R14	R1 to R4, R11 to R13 & 517	$\Rightarrow$	$\tau_{\mathrm{p_c}}\,(p_{\mathrm{c}}, \epsilon_{\mathrm{p_c},n}\,(\Delta p_{\mathrm{c}})) = \tau_{\mathrm{p_c}}\,(p_{\mathrm{c}}, n \times \Delta p_{\mathrm{c}})$ for all integer n
R15	R14 & 430	$\Rightarrow$	$\epsilon_{\mathrm{p_c},n}\,(\Delta p_{\mathrm{c}}) = n \times \Delta p_{\mathrm{c}}$

Theorem 519 *If*

$$\psi = [\mu_{\rm c}, \mu_{\rm m}, f_0, p_{{\rm c},0}]$$

is a pitch system and $\Delta p_{\rm c}$ is any chromatic pitch interval in ψ then

$$\iota_{\rm p_c}\left(\Delta p_{\rm c}\right) = \epsilon_{{\rm p_c},-1}\left(\Delta p_{\rm c}\right)$$

Proof

R1	515	$\Rightarrow$	$\iota_{\rm p_c}\left(\Delta p_{\rm c}\right) = -\Delta p_{\rm c}$
R2	518	$\Rightarrow$	$\epsilon_{{\rm p_c},-1}\left(\Delta p_{\rm c}\right) = -\Delta p_{\rm c}$
R3	R1 & R2	$\Rightarrow$	$\iota_{\rm p_c}\left(\Delta p_{\rm c}\right) = \epsilon_{{\rm p_c},-1}\left(\Delta p_{\rm c}\right)$

Theorem 520 *If*

$$\psi = [\mu_{\rm c}, \mu_{\rm m}, f_0, p_{{\rm c},0}]$$

is a pitch system, $n_1, n_2, \ldots n_k$ is a collection of integers and $\Delta p_{\rm c}$ is a chromatic pitch interval in ψ then

$$\epsilon_{{\rm p_c},n_k}\left(\ldots\epsilon_{{\rm p_c},n_2}\left(\epsilon_{{\rm p_c},n_1}\left(\Delta p_{\rm c}\right)\right)\ldots\right) = \epsilon_{{\rm p_c},\prod_{j=1}^{k} n_j}\left(\Delta p_{\rm c}\right)$$

Proof

R1	Let		$x_k = \epsilon_{\mathrm{p_c},n_k}\left(\ldots \epsilon_{\mathrm{p_c},n_2}\left(\epsilon_{\mathrm{p_c},n_1}\left(\Delta p_{\mathrm{c}}\right)\right)\ldots\right)$
R2	Let		$y_k = \epsilon_{\mathrm{p_c},\prod_{j=1}^{k} n_j}\left(\Delta p_{\mathrm{c}}\right)$
R3	R1 & R2	$\Rightarrow$	$y_1 = \epsilon_{\mathrm{p_c},\prod_{j=1}^{1} n_j}\left(\Delta p_{\mathrm{c}}\right) = \epsilon_{\mathrm{p_c},n_1}\left(\Delta p_{\mathrm{c}}\right) = x_1$
R4	R1 & R2	$\Rightarrow$	$\left(x_k = y_k \Rightarrow x_{k+1} = \epsilon_{\mathrm{p_c},n_{k+1}}\left(y_k\right)\right)$
R5	R2 & 518	$\Rightarrow$	$\epsilon_{\mathrm{p_c},n_{k+1}}\left(y_k\right) = n_{k+1} \times y_k$
			$= n_{k+1} \times \epsilon_{\mathrm{p_c},\prod_{j=1}^{k} n_j}\left(\Delta p_{\mathrm{c}}\right)$
			$= n_{k+1} \times \left(\prod_{j=1}^{k} n_j\right) \times \Delta p_{\mathrm{c}}$
			$= \left(\prod_{j=1}^{k+1} n_j\right) \times \Delta p_{\mathrm{c}}$
			$= \epsilon_{\mathrm{p_c},\prod_{j=1}^{k+1} n_j}\left(\Delta p_{\mathrm{c}}\right)$
			$= y_{k+1}$
R6	R4 & R5	$\Rightarrow$	$\left(x_k = y_k \Rightarrow x_{k+1} = y_{k+1}\right)$
R7	R3 & R6	$\Rightarrow$	$x_k = y_k$ for all integer k greater than zero
R8	R1, R2 & R7	$\Rightarrow$	$\epsilon_{\mathrm{p_c},n_k}\left(\ldots \epsilon_{\mathrm{p_c},n_2}\left(\epsilon_{\mathrm{p_c},n_1}\left(\Delta p_{\mathrm{c}}\right)\right)\ldots\right) = \epsilon_{\mathrm{p_c},\prod_{j=1}^{k} n_j}\left(\Delta p_{\mathrm{c}}\right)$

Theorem 521 *If*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

is a pitch system, n is an integer and Δp_{c} is a chromatic pitch interval in ψ then

$$\iota_{\mathrm{p_c}}\left(\epsilon_{\mathrm{p_c},n}\left(\Delta p_{\mathrm{c}}\right)\right) = \epsilon_{\mathrm{p_c},-n}\left(\Delta p_{\mathrm{c}}\right)$$

Proof

R1	515	$\Rightarrow$	$\iota_{\mathrm{p_c}}\left(\epsilon_{\mathrm{p_c},n}\left(\Delta p_{\mathrm{c}}\right)\right) = -\,\epsilon_{\mathrm{p_c},n}\left(\Delta p_{\mathrm{c}}\right)$
R2	R1 & 518	$\Rightarrow$	$\iota_{\mathrm{p_c}}\left(\epsilon_{\mathrm{p_c},n}\left(\Delta p_{\mathrm{c}}\right)\right) = -n \times \Delta p_{\mathrm{c}} = \epsilon_{\mathrm{p_c},-n}\left(\Delta p_{\mathrm{c}}\right)$

Theorem 522 *If*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

is a pitch system, n is an integer and Δp_{c} is a chromatic pitch interval in ψ then:

$$\Delta\,\mathrm{c}\left(\epsilon_{\mathrm{p_c},n}\left(\Delta p_{\mathrm{c}}\right)\right) = \epsilon_{\mathrm{c},n}\left(\Delta\,\mathrm{c}\left(\Delta p_{\mathrm{c}}\right)\right)$$

Proof

R1	Let		$x = \Delta\,\mathrm{c}\left(\epsilon_{\mathrm{p_c},n}\left(\Delta p_{\mathrm{c}}\right)\right)$
R2	Let		$y = \epsilon_{\mathrm{c},n}\left(\Delta\,\mathrm{c}\left(\Delta p_{\mathrm{c}}\right)\right)$
R3	518 & R1	$\Rightarrow$	$x = \Delta\,\mathrm{c}\left(n \times \Delta p_{\mathrm{c}}\right)$
R4	287 & R3	$\Rightarrow$	$x = \left(n \times \Delta p_{\mathrm{c}}\right) \bmod \mu_{\mathrm{c}}$
R5	R2 & 287	$\Rightarrow$	$y = \epsilon_{\mathrm{c},n}\left(\Delta p_{\mathrm{c}} \bmod \mu_{\mathrm{c}}\right)$
R6	R5 & 454	$\Rightarrow$	$y = \left(n \times \left(\Delta p_{\mathrm{c}} \bmod \mu_{\mathrm{c}}\right)\right) \bmod \mu_{\mathrm{c}}$
R7	R6 & 45	$\Rightarrow$	$y = \left(n \times \Delta p_{\mathrm{c}}\right) \bmod \mu_{\mathrm{c}}$
R8	R4 & R7	$\Rightarrow$	$x = y$
R9	R1, R2 & R8	$\Rightarrow$	$\Delta\,\mathrm{c}\left(\epsilon_{\mathrm{p_c},n}\left(\Delta p_{\mathrm{c}}\right)\right) = \epsilon_{\mathrm{c},n}\left(\Delta\,\mathrm{c}\left(\Delta p_{\mathrm{c}}\right)\right)$

Theorem 523 *If*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

is a pitch system, n is an integer and Δp_{c} is a chromatic pitch interval in ψ then:

$$\Delta\,\mathrm{f}\left(\epsilon_{\mathrm{p_c},n}\left(\Delta p_{\mathrm{c}}\right)\right) = \epsilon_{\mathrm{f},n}\left(\Delta\,\mathrm{f}\left(\Delta p_{\mathrm{c}}\right)\right)$$

Proof

R1	518	$\Rightarrow$	$\Delta\,\mathrm{f}\left(\epsilon_{\mathrm{p_c},n}\left(\Delta p_{\mathrm{c}}\right)\right) = \Delta\,\mathrm{f}\left(n \times \Delta p_{\mathrm{c}}\right)$
R2	R1 & 284	$\Rightarrow$	$\Delta\,\mathrm{f}\left(\epsilon_{\mathrm{p_c},n}\left(\Delta p_{\mathrm{c}}\right)\right) = 2^{n \times \Delta p_{\mathrm{c}}/\mu_{\mathrm{c}}}$
			$= \left(2^{\Delta p_{\mathrm{c}}/\mu_{\mathrm{c}}}\right)^n$
			$= \left(\Delta\,\mathrm{f}\left(\Delta p_{\mathrm{c}}\right)\right)^n$
R3	R2 & 549	$\Rightarrow$	$\Delta\,\mathrm{f}\left(\epsilon_{\mathrm{p_c},n}\left(\Delta p_{\mathrm{c}}\right)\right) = \epsilon_{\mathrm{f},n}\left(\Delta\,\mathrm{f}\left(\Delta p_{\mathrm{c}}\right)\right)$

Theorem 524 *If*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

is a pitch system, $n_1, n_2, \ldots n_k$ is a collection of integers and Δp_{c} is a chromatic pitch interval in ψ then

$$\sigma_{\mathrm{p_c}}\left(\epsilon_{\mathrm{p_c},n_1}\left(\Delta p_{\mathrm{c}}\right), \epsilon_{\mathrm{p_c},n_2}\left(\Delta p_{\mathrm{c}}\right), \ldots, \epsilon_{\mathrm{p_c},n_k}\left(\Delta p_{\mathrm{c}}\right)\right) = \epsilon_{\mathrm{p_c},\sum_{j=1}^{k} n_j}\left(\Delta p_{\mathrm{c}}\right)$$

Proof

R1	Let		$x = \sigma_{\mathrm{p_c}}\left(\epsilon_{\mathrm{p_c},n_1}\left(\Delta p_{\mathrm{c}}\right), \epsilon_{\mathrm{p_c},n_2}\left(\Delta p_{\mathrm{c}}\right), \ldots, \epsilon_{\mathrm{p_c},n_k}\left(\Delta p_{\mathrm{c}}\right)\right)$
R2	R1 & 511	$\Rightarrow$	$x = \sum_{j=1}^{k} \epsilon_{\mathrm{p_c},n_j}\left(\Delta p_{\mathrm{c}}\right)$
R3	R2 & 518	$\Rightarrow$	$x = \sum_{j=1}^{k}\left(n_j \times \Delta p_{\mathrm{c}}\right) = \Delta p_{\mathrm{c}} \times \sum_{j=1}^{k} n_j = \epsilon_{\mathrm{p_c},\sum_{j=1}^{k} n_j}\left(\Delta p_{\mathrm{c}}\right)$
R4	R1 & R3	$\Rightarrow$	$\sigma_{\mathrm{p_c}}\left(\epsilon_{\mathrm{p_c},n_1}\left(\Delta p_{\mathrm{c}}\right), \epsilon_{\mathrm{p_c},n_2}\left(\Delta p_{\mathrm{c}}\right), \ldots, \epsilon_{\mathrm{p_c},n_k}\left(\Delta p_{\mathrm{c}}\right)\right) = \epsilon_{\mathrm{p_c},\sum_{j=1}^{k} n_j}\left(\Delta p_{\mathrm{c}}\right)$

Exponentiation of the chromatic pitch tranposition function

Definition 525 (Definition of $\tau_{\mathrm{p_c},n}\left(p_{\mathrm{c}}, \Delta p_{\mathrm{c}}\right)$) *If ψ is a pitch system and p_{c} is a chromatic pitch in ψ and Δp_{c} is a chromatic pitch interval in ψ then*

$$\tau_{\mathrm{p_c},n}\left(p_{\mathrm{c}}, \Delta p_{\mathrm{c}}\right) = \tau_{\mathrm{p_c}}\left(p_{\mathrm{c}}, \epsilon_{\mathrm{p_c},n}\left(\Delta p_{\mathrm{c}}\right)\right)$$

Theorem 526 *If*

$$\psi = \left[\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}\right]$$

is a pitch system, $n_1, n_2, \ldots, n_k$ is a collection of integers, p_{c} is a chromatic pitch in ψ and Δp_{c} is a chromatic pitch interval in ψ then

$$\tau_{\mathrm{p_c},n_k}\left(\ldots \tau_{\mathrm{p_c},n_2}\left(\tau_{\mathrm{p_c},n_1}\left(p_{\mathrm{c}}, \Delta p_{\mathrm{c}}\right), \Delta p_{\mathrm{c}}\right) \ldots, \Delta p_{\mathrm{c}}\right) = \tau_{\mathrm{p_c},\sum_{j=1}^{k} n_j}\left(p_{\mathrm{c}}, \Delta p_{\mathrm{c}}\right)$$

Proof

R1	Let		$x_k = \tau_{\mathrm{p_c},n_k}\left(\ldots \tau_{\mathrm{p_c},n_2}\left(\tau_{\mathrm{p_c},n_1}\left(p_\mathrm{c}, \Delta p_\mathrm{c}\right), \Delta p_\mathrm{c}\right)\ldots, \Delta p_\mathrm{c}\right)$
R2	Let		$y_k = \tau_{\mathrm{p_c},\sum_{j=1}^{k} n_j}\left(p_\mathrm{c}, \Delta p_\mathrm{c}\right)$
R3	R1	$\Rightarrow$	$x_1 = \tau_{\mathrm{p_c},n_1}\left(p_\mathrm{c}, \Delta p_\mathrm{c}\right)$
R4	R2	$\Rightarrow$	$y_1 = \tau_{\mathrm{p_c},\sum_{j=1}^{1} n_j}\left(p_\mathrm{c}, \Delta p_\mathrm{c}\right) = \tau_{\mathrm{p_c},n_1}\left(p_\mathrm{c}, \Delta p_\mathrm{c}\right)$
R5	R3 & R4	$\Rightarrow$	$x_1 = y_1$
R6	R1	$\Rightarrow$	$\left(x_k = y_k \Rightarrow x_{k+1} = \tau_{\mathrm{p_c},n_{k+1}}\left(y_k, \Delta p_\mathrm{c}\right)\right)$
R7	R2	$\Rightarrow$	$\tau_{\mathrm{p_c},n_{k+1}}\left(y_k, \Delta p_\mathrm{c}\right) = \tau_{\mathrm{p_c},n_{k+1}}\left(\tau_{\mathrm{p_c},\sum_{j=1}^{k} n_j}\left(p_\mathrm{c}, \Delta p_\mathrm{c}\right), \Delta p_\mathrm{c}\right)$
R8	R7 & 525	$\Rightarrow$	$\tau_{\mathrm{p_c},n_{k+1}}\left(y_k, \Delta p_\mathrm{c}\right) = \tau_{\mathrm{p_c}}\left(\tau_{\mathrm{p_c}}\left(p_\mathrm{c}, \epsilon_{\mathrm{p_c},\sum_{j=1}^{k} n_j}\left(\Delta p_\mathrm{c}\right)\right), \epsilon_{\mathrm{p_c},n_{k+1}}\left(\Delta p_\mathrm{c}\right)\right)$
R9	R8 & 518	$\Rightarrow$	$\tau_{\mathrm{p_c},n_{k+1}}\left(y_k, \Delta p_\mathrm{c}\right) = \tau_{\mathrm{p_c}}\left(\tau_{\mathrm{p_c}}\left(p_\mathrm{c}, \left(\sum_{j=1}^{k} n_j\right) \times \Delta p_\mathrm{c}\right), n_{k+1} \times \Delta p_\mathrm{c}\right)$
R10	R9 & 427	$\Rightarrow$	$\tau_{\mathrm{p_c},n_{k+1}}\left(y_k, \Delta p_\mathrm{c}\right) = p_\mathrm{c} + \left(\sum_{j=1}^{k} n_j\right) \times \Delta p_\mathrm{c} + n_{k+1} \times \Delta p_\mathrm{c}$
			$= p_\mathrm{c} + \Delta p_\mathrm{c} \times \left(n_{k+1} + \sum_{j=1}^{k} n_j\right)$
			$= p_\mathrm{c} + \Delta p_\mathrm{c} \times \sum_{j=1}^{k+1} n_j$
			$= \tau_{\mathrm{p_c}}\left(p_\mathrm{c}, \Delta p_\mathrm{c} \times \sum_{j=1}^{k+1} n_j\right)$
R11	R10 & 518	$\Rightarrow$	$\tau_{\mathrm{p_c},n_{k+1}}\left(y_k, \Delta p_\mathrm{c}\right) = \tau_{\mathrm{p_c}}\left(p_\mathrm{c}, \epsilon_{\mathrm{p_c},\sum_{j=1}^{k+1} n_j}\left(\Delta p_\mathrm{c}\right)\right)$
R12	R11 & 525	$\Rightarrow$	$\tau_{\mathrm{p_c},n_{k+1}}\left(y_k, \Delta p_\mathrm{c}\right) = \tau_{\mathrm{p_c},\sum_{j=1}^{k+1} n_j}\left(p_\mathrm{c}, \Delta p_\mathrm{c}\right)$
R13	R2 & R12	$\Rightarrow$	$\tau_{\mathrm{p_c},n_{k+1}}\left(y_k, \Delta p_\mathrm{c}\right) = y_{k+1}$
R14	R6 & R13	$\Rightarrow$	$\left(x_k = y_k \Rightarrow x_{k+1} = y_{k+1}\right)$
R15	R5 & R14	$\Rightarrow$	$x_k = y_k$ for all integer k greater than zero
R16	R1, R2 & R15	$\Rightarrow$	$\tau_{\mathrm{p_c},n_k}\left(\ldots \tau_{\mathrm{p_c},n_2}\left(\tau_{\mathrm{p_c},n_1}\left(p_\mathrm{c}, \Delta p_\mathrm{c}\right), \Delta p_\mathrm{c}\right)\ldots, \Delta p_\mathrm{c}\right) = \tau_{\mathrm{p_c},\sum_{j=1}^{k} n_j}\left(p_\mathrm{c}, \Delta p_\mathrm{c}\right)$

4.6.6 Summation, inversion and exponentiation of morphetic pitch intervals

Summation of morphetic pitch intervals

Definition 527 *If*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

is a pitch system and

$$\Delta p_{\mathrm{m},1}, \Delta p_{\mathrm{m},2}, \ldots \Delta p_{\mathrm{m},n}$$

is a collection of morphetic pitch intervals in ψ *then*

$$\sigma_{\mathrm{p_m}} \left(\Delta p_{\mathrm{m},1}, \Delta p_{\mathrm{m},2}, \ldots \Delta p_{\mathrm{m},n} \right) = \sum_{k=1}^{n} \Delta p_{\mathrm{m},k}$$

Theorem 528 *If* ψ *is a pitch system and*

$$\Delta p_{\mathrm{m},1}, \Delta p_{\mathrm{m},2}, \ldots \Delta p_{\mathrm{m},n}$$

is a collection of morphetic pitch intervals in ψ *and* p_{m} *is a morphetic pitch in* ψ *then*

$$\tau_{\mathrm{p_m}} \left(p_{\mathrm{m}}, \sigma_{\mathrm{p_m}} \left(\Delta p_{\mathrm{m},1}, \Delta p_{\mathrm{m},2}, \ldots \Delta p_{\mathrm{m},n} \right) \right) = \tau_{\mathrm{p_m}} \left(\ldots \tau_{\mathrm{p_m}} \left(\tau_{\mathrm{p_m}} \left(p_{\mathrm{m}}, \Delta p_{\mathrm{m},1} \right), \Delta p_{\mathrm{m},2} \right) \ldots, \Delta p_{\mathrm{m},n} \right)$$

Proof

R1	Let		$x_n = \tau_{\mathrm{p_m}}\left(\ldots \tau_{\mathrm{p_m}}\left(\tau_{\mathrm{p_m}}\left(p_{\mathrm{m}}, \Delta p_{\mathrm{m},1}\right), \Delta p_{\mathrm{m},2}\right)\ldots, \Delta p_{\mathrm{m},n}\right)$
R2	Let		$y_n = \tau_{\mathrm{p_m}}\left(p_{\mathrm{m}}, \sigma_{\mathrm{p_m}}\left(\Delta p_{\mathrm{m},1}, \Delta p_{\mathrm{m},2}, \ldots \Delta p_{\mathrm{m},n}\right)\right)$
R3	R1 & 432	$\Rightarrow$	$x_1 = \tau_{\mathrm{p_m}}\left(p_{\mathrm{m}}, \Delta p_{\mathrm{m},1}\right) = p_{\mathrm{m}} + \Delta p_{\mathrm{m},1}$
R4	R2 & 432	$\Rightarrow$	$y_1 = p_{\mathrm{m}} + \sigma_{\mathrm{p_m}}\left(\Delta p_{\mathrm{m},1}\right)$
R5	R4 & 527	$\Rightarrow$	$y_1 = p_{\mathrm{m}} + \Delta p_{\mathrm{m},1}$
R6	R3 & R5	$\Rightarrow$	$x_1 = y_1$
R7	R1	$\Rightarrow$	$\left(x_k = y_k \Rightarrow x_{k+1} = \tau_{\mathrm{p_m}}\left(y_k, \Delta p_{\mathrm{m},k+1}\right)\right)$
R8	R2 & 432	$\Rightarrow$	$y_n = p_{\mathrm{m}} + \sigma_{\mathrm{p_m}}\left(\Delta p_{\mathrm{m},1}, \Delta p_{\mathrm{m},2}, \ldots \Delta p_{\mathrm{m},n}\right)$
R9	R8 & 527	$\Rightarrow$	$y_n = p_{\mathrm{m}} + \sum_{k=1}^{n} \Delta p_{\mathrm{m},k}$
R10	432	$\Rightarrow$	$\tau_{\mathrm{p_m}}\left(y_k, \Delta p_{\mathrm{m},k+1}\right) = y_k + \Delta p_{\mathrm{m},k+1}$
R11	R9 & R10	$\Rightarrow$	$\tau_{\mathrm{p_m}}\left(y_k, \Delta p_{\mathrm{m},k+1}\right) = p_{\mathrm{m}} + \sum_{j=1}^{k} \Delta p_{\mathrm{m},j} + \Delta p_{\mathrm{m},k+1}$
			$= p_{\mathrm{m}} + \sum_{j=1}^{k+1} \Delta p_{\mathrm{m},j}$
R12	R11 & R9	$\Rightarrow$	$\tau_{\mathrm{p_m}}\left(y_k, \Delta p_{\mathrm{m},k+1}\right) = y_{k+1}$
R13	R12 & R7	$\Rightarrow$	$\left(x_k = y_k \Rightarrow x_{k+1} = y_{k+1}\right)$
R14	R6 & R13	$\Rightarrow$	$x_k = y_k$ for all positive integers k
R15	R1, R2 & R14	$\Rightarrow$	$\tau_{\mathrm{p_m}}\left(p_{\mathrm{m}}, \sigma_{\mathrm{p_m}}\left(\Delta p_{\mathrm{m},1}, \Delta p_{\mathrm{m},2}, \ldots \Delta p_{\mathrm{m},n}\right)\right) = \tau_{\mathrm{p_m}}\left(\ldots \tau_{\mathrm{p_m}}\left(\tau_{\mathrm{p_m}}\left(p_{\mathrm{m}}, \Delta p_{\mathrm{m},1}\right), \Delta p_{\mathrm{m},2}\right)\ldots, \Delta p_{\mathrm{m},n}\right)$

Inversion of morphetic pitch intervals

Definition 529 (Definition of $\iota_{\mathrm{p_m}}\left(\Delta p_{\mathrm{m}}\right)$) *If ψ is a pitch system and Δp_{m} is a morphetic pitch interval in ψ and p_{m} is a morphetic pitch in ψ then $\iota_{\mathrm{p_m}}\left(\Delta p_{\mathrm{m}}\right)$ is the morphetic pitch interval that satisfies the following equation*

$$\tau_{\mathrm{p_m}}\left(\tau_{\mathrm{p_m}}\left(p_{\mathrm{m}}, \Delta p_{\mathrm{m}}\right), \iota_{\mathrm{p_m}}\left(\Delta p_{\mathrm{m}}\right)\right) = p_{\mathrm{m}}$$

Definition 530 (Inversional equivalence of morphetic pitch intervals) *If ψ is a pitch system and $\Delta p_{\mathrm{m},1}$ and $\Delta p_{\mathrm{m},2}$ are morphetic pitch intervals in ψ then $\Delta p_{\mathrm{m},1}$ and $\Delta p_{\mathrm{m},2}$ are* inversionally equivalent *if and only if*

$$\left(\iota_{\mathrm{p_m}}\left(\Delta p_{\mathrm{m},1}\right) = \Delta p_{\mathrm{m},2}\right) \vee \left(\Delta p_{\mathrm{m},1} = \Delta p_{\mathrm{m},2}\right)$$

The fact that two morphetic pitch intervals are inversionally equivalent is denoted as follows:

$$\Delta p_{\mathrm{m},1} \equiv_{\iota} \Delta p_{\mathrm{m},2}$$

Theorem 531 *If*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

is a pitch system and Δp_{m} *is a morphetic pitch interval in* ψ *then*

$$\iota_{\mathrm{p_m}}(\Delta p_{\mathrm{m}}) = -\Delta p_{\mathrm{m}}$$

Proof

R1	529	$\Rightarrow$	$\tau_{\mathrm{p_m}}(\tau_{\mathrm{p_m}}(p_{\mathrm{m}}, \Delta p_{\mathrm{m}}), \iota_{\mathrm{p_m}}(\Delta p_{\mathrm{m}})) = p_{\mathrm{m}}$
R2	R1 & 432	$\Rightarrow$	$\tau_{\mathrm{p_m}}(p_{\mathrm{m}} + \Delta p_{\mathrm{m}}, \iota_{\mathrm{p_m}}(\Delta p_{\mathrm{m}})) = p_{\mathrm{m}}$
		$\Rightarrow$	$p_{\mathrm{m}} + \Delta p_{\mathrm{m}} + \iota_{\mathrm{p_m}}(\Delta p_{\mathrm{m}}) = p_{\mathrm{m}}$
		$\Rightarrow$	$\Delta p_{\mathrm{m}} + \iota_{\mathrm{p_m}}(\Delta p_{\mathrm{m}}) = 0$
		$\Rightarrow$	$\iota_{\mathrm{p_m}}(\Delta p_{\mathrm{m}}) = -\Delta p_{\mathrm{m}}$

Theorem 532 *If* ψ *is a pitch system and* Δp_{m}*,* $\Delta p_{\mathrm{m},1}$ *and* $\Delta p_{\mathrm{m},2}$ *are morphetic pitch intervals in* ψ *then*

$$(\Delta p_{\mathrm{m},1} = \iota_{\mathrm{p_m}}(\Delta p_{\mathrm{m}})) \wedge (\Delta p_{\mathrm{m},2} = \iota_{\mathrm{p_m}}(\Delta p_{\mathrm{m}})) \Rightarrow (\Delta p_{\mathrm{m},1} = \Delta p_{\mathrm{m},2})$$

Proof

R1	Let		$(\Delta p_{\mathrm{m},1} = \iota_{\mathrm{p_m}}(\Delta p_{\mathrm{m}})) \wedge (\Delta p_{\mathrm{m},2} = \iota_{\mathrm{p_m}}(\Delta p_{\mathrm{m}}))$
R2	R1 & 531	$\Rightarrow$	$\Delta p_{\mathrm{m},1} = -\Delta p_{\mathrm{m}}$
R3	R1 & 531	$\Rightarrow$	$\Delta p_{\mathrm{m},2} = -\Delta p_{\mathrm{m}}$
R4	R2 & R3	$\Rightarrow$	$\Delta p_{\mathrm{m},1} = \Delta p_{\mathrm{m},2}$
R5	R1 to R4	$\Rightarrow$	$(\Delta p_{\mathrm{m},1} = \iota_{\mathrm{p_m}}(\Delta p_{\mathrm{m}})) \wedge (\Delta p_{\mathrm{m},2} = \iota_{\mathrm{p_m}}(\Delta p_{\mathrm{m}})) \Rightarrow (\Delta p_{\mathrm{m},1} = \Delta p_{\mathrm{m},2})$

Exponentiation of morphetic pitch intervals

Definition 533 (Definition of $\epsilon_{\mathrm{p_m},n}(\Delta p_{\mathrm{m}})$) *Given that:*

1. ψ *is a pitch system;*
2. p_{m} *is a morphetic pitch in* ψ*;*
3. Δp_{m} *is a morphetic pitch interval in* ψ*;*
4. n *is an integer;*

5. *k is an integer and $1 \leq k \leq \mathrm{abs}(n)$;*

6. *$\Delta p_{\mathrm{m},1,k} = \Delta p_{\mathrm{m}}$ for all k; and*

7. *$\Delta p_{\mathrm{m},2,k} = \iota_{\mathrm{p_m}}(\Delta p_{\mathrm{m}})$ for all k;*

then $\epsilon_{\mathrm{p_m},n}(\Delta p_{\mathrm{m}})$ returns a morphetic pitch interval that satisfies the following equation:

$$\tau_{\mathrm{p_m}}(p_{\mathrm{m}}, \epsilon_{\mathrm{p_m},n}(\Delta p_{\mathrm{m}})) = \begin{cases} \tau_{\mathrm{p_m}}(p_{\mathrm{m}}, \sigma_{\mathrm{p_m}}(\Delta p_{\mathrm{m},1,1}, \Delta p_{\mathrm{m},1,2}, \ldots \Delta p_{\mathrm{m},1,n})) & \text{if} \quad n > 0 \\ p_{\mathrm{m}} & \text{if} \quad n = 0 \\ \tau_{\mathrm{p_m}}(p_{\mathrm{m}}, \sigma_{\mathrm{p_m}}(\Delta p_{\mathrm{m},2,1}, \Delta p_{\mathrm{m},2,2}, \ldots \Delta p_{\mathrm{m},2,-n})) & \text{if} \quad n < 0 \end{cases}$$

Theorem 534 (Formula for $\epsilon_{\mathrm{p_m},n}(\Delta p_{\mathrm{m}})$) *If*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

is a pitch system and Δp_{m} is a morphetic pitch interval in ψ and n is an integer then

$$\epsilon_{\mathrm{p_m},n}(\Delta p_{\mathrm{m}}) = n \times \Delta p_{\mathrm{m}}$$

Proof

R1	Let		n be any integer
R2	Let		k be an integer such that $1 \leq k \leq \mathrm{abs}\,(n)$
R3	Let		$\Delta p_{\mathrm{m},1,k} = \Delta p_{\mathrm{m}}$ for all k
R4	Let		$\Delta p_{\mathrm{m},2,k} = \iota_{\mathrm{p_m}}\,(\Delta p_{\mathrm{m}})$ for all k
R5	Let		n_1 be any integer greater than zero
R6	R3, R5 & 527	$\Rightarrow$	$\tau_{\mathrm{p_m}}\,(p_{\mathrm{m}}, \sigma_{\mathrm{p_m}}\,(\Delta p_{\mathrm{m},1,1}, \Delta p_{\mathrm{m},1,2}, \ldots \Delta p_{\mathrm{m},1,n_1}))$
			$= \tau_{\mathrm{p_m}}\left(p_{\mathrm{m}}, \sum_{j=1}^{n_1} \Delta p_{\mathrm{m},1,j}\right)$
			$= \tau_{\mathrm{p_m}}\,(p_{\mathrm{m}}, n_1 \times \Delta p_{\mathrm{m}})$
R7	432	$\Rightarrow$	$\tau_{\mathrm{p_m}}\,(p_{\mathrm{m}}, 0 \times \Delta p_{\mathrm{m}}) = p_{\mathrm{m}} + 0 \times \Delta p_{\mathrm{m}} = p_{\mathrm{m}}$
R8	Let		n_2 be any integer less than zero
R9	R4, R8 & 527	$\Rightarrow$	$\tau_{\mathrm{p_m}}\,(p_{\mathrm{m}}, \sigma_{\mathrm{p_m}}\,(\Delta p_{\mathrm{m},2,1}, \Delta p_{\mathrm{m},2,2}, \ldots \Delta p_{\mathrm{m},2,-n_2}))$
			$= \tau_{\mathrm{p_m}}\left(p_{\mathrm{m}}, \sum_{j=1}^{-n_2} \Delta p_{\mathrm{m},2,j}\right)$
			$= \tau_{\mathrm{p_m}}\,(p_{\mathrm{m}}, -n_2 \times \iota_{\mathrm{p_m}}\,(\Delta p_{\mathrm{m}}))$
R10	R9 & 531	$\Rightarrow$	$\tau_{\mathrm{p_m}}\,(p_{\mathrm{m}}, \sigma_{\mathrm{p_m}}\,(\Delta p_{\mathrm{m},2,1}, \Delta p_{\mathrm{m},2,2}, \ldots \Delta p_{\mathrm{m},2,-n_2}))$
			$= \tau_{\mathrm{p_m}}\,(p_{\mathrm{m}}, -n_2 \times (-\Delta p_{\mathrm{m}}))$
			$= \tau_{\mathrm{p_m}}\,(p_{\mathrm{m}}, n_2 \times \Delta p_{\mathrm{m}})$
R11	R1, R5 & R6	$\Rightarrow$	$\tau_{\mathrm{p_m}}\,(p_{\mathrm{m}}, \sigma_{\mathrm{p_m}}\,(\Delta p_{\mathrm{m},1,1}, \Delta p_{\mathrm{m},1,2}, \ldots \Delta p_{\mathrm{m},1,n})) = \tau_{\mathrm{p_m}}\,(p_{\mathrm{m}}, n \times \Delta p_{\mathrm{m}})$ when $n > 0$
R12	R1 & R7	$\Rightarrow$	$p_{\mathrm{m}} = \tau_{\mathrm{p_m}}\,(p_{\mathrm{m}}, n \times \Delta p_{\mathrm{m}})$ when $n = 0$
R13	R1, R8 & R10	$\Rightarrow$	$\tau_{\mathrm{p_m}}\,(p_{\mathrm{m}}, \sigma_{\mathrm{p_m}}\,(\Delta p_{\mathrm{m},2,1}, \Delta p_{\mathrm{m},2,2}, \ldots \Delta p_{\mathrm{m},2,-n})) = \tau_{\mathrm{p_m}}\,(p_{\mathrm{m}}, n \times \Delta p_{\mathrm{m}})$ when $n < 0$
R14	R1 to R4, R11 to R13 & 533	$\Rightarrow$	$\tau_{\mathrm{p_m}}\,(p_{\mathrm{m}}, \epsilon_{\mathrm{p_m},n}\,(\Delta p_{\mathrm{m}})) = \tau_{\mathrm{p_m}}\,(p_{\mathrm{m}}, n \times \Delta p_{\mathrm{m}})$ for all integer n
R15	R14 & 435	$\Rightarrow$	$\epsilon_{\mathrm{p_m},n}\,(\Delta p_{\mathrm{m}}) = n \times \Delta p_{\mathrm{m}}$

Theorem 535 *If*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

is a pitch system and Δp_{m} *is any morphetic pitch interval in* ψ *then*

$$\iota_{\mathrm{p_m}}\left(\Delta p_{\mathrm{m}}\right) = \epsilon_{\mathrm{p_m},-1}\left(\Delta p_{\mathrm{m}}\right)$$

Proof

R1	531	$\Rightarrow$	$\iota_{\mathrm{p_m}}\left(\Delta p_{\mathrm{m}}\right) = -\Delta p_{\mathrm{m}}$
R2	534	$\Rightarrow$	$\epsilon_{\mathrm{p_m},-1}\left(\Delta p_{\mathrm{m}}\right) = -\Delta p_{\mathrm{m}}$
R3	R1 & R2	$\Rightarrow$	$\iota_{\mathrm{p_m}}\left(\Delta p_{\mathrm{m}}\right) = \epsilon_{\mathrm{p_m},-1}\left(\Delta p_{\mathrm{m}}\right)$

Theorem 536 *If*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

is a pitch system, $n_1, n_2, \ldots n_k$ *is a collection of integers and* Δp_{m} *is a morphetic pitch interval in* ψ *then*

$$\epsilon_{\mathrm{p_m},n_k}\left(\ldots \epsilon_{\mathrm{p_m},n_2}\left(\epsilon_{\mathrm{p_m},n_1}\left(\Delta p_{\mathrm{m}}\right)\right)\ldots\right) = \epsilon_{\mathrm{p_m},\prod_{j=1}^{k} n_j}\left(\Delta p_{\mathrm{m}}\right)$$

Proof

R1	Let		$x_k = \epsilon_{\mathrm{p_m},n_k}\left(\ldots\epsilon_{\mathrm{p_m},n_2}\left(\epsilon_{\mathrm{p_m},n_1}\left(\Delta p_\mathrm{m}\right)\right)\ldots\right)$
R2	Let		$y_k = \epsilon_{\mathrm{p_m},\prod_{j=1}^{k} n_j}\left(\Delta p_\mathrm{m}\right)$
R3	R1 & R2	$\Rightarrow$	$y_1 = \epsilon_{\mathrm{p_m},\prod_{j=1}^{1} n_j}\left(\Delta p_\mathrm{m}\right) = \epsilon_{\mathrm{p_m},n_1}\left(\Delta p_\mathrm{m}\right) = x_1$
R4	R1 & R2	$\Rightarrow$	$\left(x_k = y_k \Rightarrow x_{k+1} = \epsilon_{\mathrm{p_m},n_{k+1}}\left(y_k\right)\right)$
R5	R2 & 534	$\Rightarrow$	$\epsilon_{\mathrm{p_m},n_{k+1}}\left(y_k\right) = n_{k+1} \times y_k$
			$= n_{k+1} \times \epsilon_{\mathrm{p_m},\prod_{j=1}^{k} n_j}\left(\Delta p_\mathrm{m}\right)$
			$= n_{k+1} \times \left(\prod_{j=1}^{k} n_j\right) \times \Delta p_\mathrm{m}$
			$= \left(\prod_{j=1}^{k+1} n_j\right) \times \Delta p_\mathrm{m}$
			$= \epsilon_{\mathrm{p_m},\prod_{j=1}^{k+1} n_j}\left(\Delta p_\mathrm{m}\right)$
			$= y_{k+1}$
R6	R4 & R5	$\Rightarrow$	$\left(x_k = y_k \Rightarrow x_{k+1} = y_{k+1}\right)$
R7	R3 & R6	$\Rightarrow$	$x_k = y_k$ for all integer k greater than zero
R8	R1, R2 & R7	$\Rightarrow$	$\epsilon_{\mathrm{p_m},n_k}\left(\ldots\epsilon_{\mathrm{p_m},n_2}\left(\epsilon_{\mathrm{p_m},n_1}\left(\Delta p_\mathrm{m}\right)\right)\ldots\right) = \epsilon_{\mathrm{p_m},\prod_{j=1}^{k} n_j}\left(\Delta p_\mathrm{m}\right)$

Theorem 537 *If*

$$\psi = [\mu_\mathrm{c}, \mu_\mathrm{m}, f_0, p_{\mathrm{c},0}]$$

is a pitch system, n is an integer and Δp_m is a morphetic pitch interval in ψ then

$$\iota_{\mathrm{p_m}}\left(\epsilon_{\mathrm{p_m},n}\left(\Delta p_\mathrm{m}\right)\right) = \epsilon_{\mathrm{p_m},-n}\left(\Delta p_\mathrm{m}\right)$$

Proof

R1	531	$\Rightarrow$	$\iota_{\mathrm{p_m}}\left(\epsilon_{\mathrm{p_m},n}\left(\Delta p_\mathrm{m}\right)\right) = -\,\epsilon_{\mathrm{p_m},n}\left(\Delta p_\mathrm{m}\right)$
R2	R1 & 534	$\Rightarrow$	$\iota_{\mathrm{p_m}}\left(\epsilon_{\mathrm{p_m},n}\left(\Delta p_\mathrm{m}\right)\right) = -n \times \Delta p_\mathrm{m} = \epsilon_{\mathrm{p_m},-n}\left(\Delta p_\mathrm{m}\right)$

Theorem 538 *If*

$$\psi = [\mu_\mathrm{c}, \mu_\mathrm{m}, f_0, p_{\mathrm{c},0}]$$

is a pitch system, n is an integer and Δp_m is a morphetic pitch interval in ψ then:

$$\Delta\,\mathrm{m}\left(\epsilon_{\mathrm{p_m},n}\left(\Delta p_\mathrm{m}\right)\right) = \epsilon_{\mathrm{m},n}\left(\Delta\,\mathrm{m}\left(\Delta p_\mathrm{m}\right)\right)$$

Proof

R1	Let		$x = \Delta\,\mathrm{m}\left(\epsilon_{\mathrm{p_m},n}\left(\Delta p_\mathrm{m}\right)\right)$
R2	Let		$y = \epsilon_{\mathrm{m},n}\left(\Delta\,\mathrm{m}\left(\Delta p_\mathrm{m}\right)\right)$
R3	534 & R1	$\Rightarrow$	$x = \Delta\,\mathrm{m}\left(n \times \Delta p_\mathrm{m}\right)$
R4	290 & R3	$\Rightarrow$	$x = \left(n \times \Delta p_\mathrm{m}\right) \bmod \mu_\mathrm{m}$
R5	R2 & 290	$\Rightarrow$	$y = \epsilon_{\mathrm{m},n}\left(\Delta p_\mathrm{m} \bmod \mu_\mathrm{m}\right)$
R6	R5 & 468	$\Rightarrow$	$y = \left(n \times \left(\Delta p_\mathrm{m} \bmod \mu_\mathrm{m}\right)\right) \bmod \mu_\mathrm{m}$
R7	R6 & 45	$\Rightarrow$	$y = \left(n \times \Delta p_\mathrm{m}\right) \bmod \mu_\mathrm{m}$
R8	R4 & R7	$\Rightarrow$	$x = y$
R9	R1, R2 & R8	$\Rightarrow$	$\Delta\,\mathrm{m}\left(\epsilon_{\mathrm{p_m},n}\left(\Delta p_\mathrm{m}\right)\right) = \epsilon_{\mathrm{m},n}\left(\Delta\,\mathrm{m}\left(\Delta p_\mathrm{m}\right)\right)$

Theorem 539 *If*

$$\psi = [\mu_\mathrm{c}, \mu_\mathrm{m}, f_0, p_{\mathrm{c},0}]$$

is a pitch system, $n_1, n_2, \ldots n_k$ is a collection of integers and Δp_m is a morphetic pitch interval in ψ then

$$\sigma_{\mathrm{p_m}}\left(\epsilon_{\mathrm{p_m},n_1}\left(\Delta p_\mathrm{m}\right), \epsilon_{\mathrm{p_m},n_2}\left(\Delta p_\mathrm{m}\right), \ldots, \epsilon_{\mathrm{p_m},n_k}\left(\Delta p_\mathrm{m}\right)\right) = \epsilon_{\mathrm{p_m},\sum_{j=1}^{k} n_j}\left(\Delta p_\mathrm{m}\right)$$

Proof

R1	Let		$x = \sigma_{\mathrm{p_m}}\left(\epsilon_{\mathrm{p_m},n_1}\left(\Delta p_\mathrm{m}\right), \epsilon_{\mathrm{p_m},n_2}\left(\Delta p_\mathrm{m}\right), \ldots, \epsilon_{\mathrm{p_m},n_k}\left(\Delta p_\mathrm{m}\right)\right)$
R2	R1 & 527	$\Rightarrow$	$x = \sum_{j=1}^{k} \epsilon_{\mathrm{p_m},n_j}\left(\Delta p_\mathrm{m}\right)$
R3	R2 & 534	$\Rightarrow$	$x = \sum_{j=1}^{k}\left(n_j \times \Delta p_\mathrm{m}\right) = \Delta p_\mathrm{m} \times \sum_{j=1}^{k} n_j = \epsilon_{\mathrm{p_m},\sum_{j=1}^{k} n_j}\left(\Delta p_\mathrm{m}\right)$
R4	R1 & R3	$\Rightarrow$	$\sigma_{\mathrm{p_m}}\left(\epsilon_{\mathrm{p_m},n_1}\left(\Delta p_\mathrm{m}\right), \epsilon_{\mathrm{p_m},n_2}\left(\Delta p_\mathrm{m}\right), \ldots, \epsilon_{\mathrm{p_m},n_k}\left(\Delta p_\mathrm{m}\right)\right) = \epsilon_{\mathrm{p_m},\sum_{j=1}^{k} n_j}\left(\Delta p_\mathrm{m}\right)$

Exponentiation of the morphetic pitch tranposition function

Definition 540 (Definition of $\tau_{\mathrm{p_m},n}\left(p_\mathrm{m}, \Delta p_\mathrm{m}\right)$) *If ψ is a pitch system and p_m is a morphetic pitch in ψ and Δp_m is a morphetic pitch interval in ψ then*

$$\tau_{\mathrm{p_m},n}\left(p_\mathrm{m}, \Delta p_\mathrm{m}\right) = \tau_{\mathrm{p_m}}\left(p_\mathrm{m}, \epsilon_{\mathrm{p_m},n}\left(\Delta p_\mathrm{m}\right)\right)$$

Theorem 541 *If*

$$\psi = [\mu_\mathrm{c}, \mu_\mathrm{m}, f_0, p_{\mathrm{c},0}]$$

is a pitch system, $n_1, n_2, \ldots, n_k$ is a collection of integers, p_m is a morphetic pitch in ψ and Δp_m is a morphetic pitch interval in ψ then

$$\tau_{\mathrm{p_m},n_k}\left(\ldots\tau_{\mathrm{p_m},n_2}\left(\tau_{\mathrm{p_m},n_1}\left(p_\mathrm{m},\Delta p_\mathrm{m}\right),\Delta p_\mathrm{m}\right)\ldots,\Delta p_\mathrm{m}\right)=\tau_{\mathrm{p_m},\sum_{j=1}^{k}n_j}\left(p_\mathrm{m},\Delta p_\mathrm{m}\right)$$

Proof

R1	Let		$x_k=\tau_{\mathrm{p_m},n_k}\left(\ldots\tau_{\mathrm{p_m},n_2}\left(\tau_{\mathrm{p_m},n_1}\left(p_\mathrm{m},\Delta p_\mathrm{m}\right),\Delta p_\mathrm{m}\right)\ldots,\Delta p_\mathrm{m}\right)$
R2	Let		$y_k=\tau_{\mathrm{p_m},\sum_{j=1}^{k}n_j}\left(p_\mathrm{m},\Delta p_\mathrm{m}\right)$
R3	R1	$\Rightarrow$	$x_1=\tau_{\mathrm{p_m},n_1}\left(p_\mathrm{m},\Delta p_\mathrm{m}\right)$
R4	R2	$\Rightarrow$	$y_1=\tau_{\mathrm{p_m},\sum_{j=1}^{1}n_j}\left(p_\mathrm{m},\Delta p_\mathrm{m}\right)=\tau_{\mathrm{p_m},n_1}\left(p_\mathrm{m},\Delta p_\mathrm{m}\right)$
R5	R3 & R4	$\Rightarrow$	$x_1=y_1$
R6	R1	$\Rightarrow$	$\left(x_k=y_k\Rightarrow x_{k+1}=\tau_{\mathrm{p_m},n_{k+1}}\left(y_k,\Delta p_\mathrm{m}\right)\right)$
R7	R2	$\Rightarrow$	$\tau_{\mathrm{p_m},n_{k+1}}\left(y_k,\Delta p_\mathrm{m}\right)=\tau_{\mathrm{p_m},n_{k+1}}\left(\tau_{\mathrm{p_m},\sum_{j=1}^{k}n_j}\left(p_\mathrm{m},\Delta p_\mathrm{m}\right),\Delta p_\mathrm{m}\right)$
R8	R7 & 540	$\Rightarrow$	$\tau_{\mathrm{p_m},n_{k+1}}\left(y_k,\Delta p_\mathrm{m}\right)=\tau_{\mathrm{p_m}}\left(\tau_{\mathrm{p_m}}\left(p_\mathrm{m},\epsilon_{\mathrm{p_m},\sum_{j=1}^{k}n_j}\left(\Delta p_\mathrm{m}\right)\right),\epsilon_{\mathrm{p_m},n_{k+1}}\left(\Delta p_\mathrm{m}\right)\right)$
R9	R8 & 534	$\Rightarrow$	$\tau_{\mathrm{p_m},n_{k+1}}\left(y_k,\Delta p_\mathrm{m}\right)=\tau_{\mathrm{p_m}}\left(\tau_{\mathrm{p_m}}\left(p_\mathrm{m},\left(\sum_{j=1}^{k}n_j\right)\times\Delta p_\mathrm{m}\right),n_{k+1}\times\Delta p_\mathrm{m}\right)$
R10	R9 & 432	$\Rightarrow$	$\tau_{\mathrm{p_m},n_{k+1}}\left(y_k,\Delta p_\mathrm{m}\right)=p_\mathrm{m}+\left(\sum_{j=1}^{k}n_j\right)\times\Delta p_\mathrm{m}+n_{k+1}\times\Delta p_\mathrm{m}$
			$=p_\mathrm{m}+\Delta p_\mathrm{m}\times\left(n_{k+1}+\sum_{j=1}^{k}n_j\right)$
			$=p_\mathrm{m}+\Delta p_\mathrm{m}\times\sum_{j=1}^{k+1}n_j$
			$=\tau_{\mathrm{p_m}}\left(p_\mathrm{m},\Delta p_\mathrm{m}\times\sum_{j=1}^{k+1}n_j\right)$
R11	R10 & 534	$\Rightarrow$	$\tau_{\mathrm{p_m},n_{k+1}}\left(y_k,\Delta p_\mathrm{m}\right)=\tau_{\mathrm{p_m}}\left(p_\mathrm{m},\epsilon_{\mathrm{p_m},\sum_{j=1}^{k+1}n_j}\left(\Delta p_\mathrm{m}\right)\right)$
R12	R11 & 540	$\Rightarrow$	$\tau_{\mathrm{p_m},n_{k+1}}\left(y_k,\Delta p_\mathrm{m}\right)=\tau_{\mathrm{p_m},\sum_{j=1}^{k+1}n_j}\left(p_\mathrm{m},\Delta p_\mathrm{m}\right)$
R13	R2 & R12	$\Rightarrow$	$\tau_{\mathrm{p_m},n_{k+1}}\left(y_k,\Delta p_\mathrm{m}\right)=y_{k+1}$
R14	R6 & R13	$\Rightarrow$	$\left(x_k=y_k\Rightarrow x_{k+1}=y_{k+1}\right)$
R15	R5 & R14	$\Rightarrow$	$x_k=y_k$ for all integer k greater than zero
R16	R1, R2 & R15	$\Rightarrow$	$\tau_{\mathrm{p_m},n_k}\left(\ldots\tau_{\mathrm{p_m},n_2}\left(\tau_{\mathrm{p_m},n_1}\left(p_\mathrm{m},\Delta p_\mathrm{m}\right),\Delta p_\mathrm{m}\right)\ldots,\Delta p_\mathrm{m}\right)=\tau_{\mathrm{p_m},\sum_{j=1}^{k}n_j}\left(p_\mathrm{m},\Delta p_\mathrm{m}\right)$

4.6.7 Summation, inversion and exponentiation of frequency intervals

Summation of frequency intervals

Definition 542 *If*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

is a pitch system and

$$\Delta f_1, \Delta f_2, \ldots \Delta f_n$$

is a collection of frequency intervals in ψ *then*

$$\sigma_{\mathrm{f}}\left(\Delta f_1, \Delta f_2, \ldots \Delta f_n\right) = \prod_{k=1}^{n} \Delta f_k$$

Theorem 543 *If* ψ *is a pitch system and*

$$\Delta f_1, \Delta f_2, \ldots \Delta f_n$$

is a collection of frequency intervals in ψ *and* f *is a frequency in* ψ *then*

$$\tau_{\mathrm{f}}\left(f, \sigma_{\mathrm{f}}\left(\Delta f_1, \Delta f_2, \ldots \Delta f_n\right)\right) = \tau_{\mathrm{f}}\left(\ldots \tau_{\mathrm{f}}\left(\tau_{\mathrm{f}}\left(f, \Delta f_1\right), \Delta f_2\right) \ldots, \Delta f_n\right)$$

Proof

R1	Let		$x_n = \tau_{\mathrm{f}}\left(f, \sigma_{\mathrm{f}}\left(\Delta f_1, \Delta f_2, \ldots \Delta f_n\right)\right)$
R2	Let		$y_n = \tau_{\mathrm{f}}\left(\ldots \tau_{\mathrm{f}}\left(\tau_{\mathrm{f}}\left(f, \Delta f_1\right), \Delta f_2\right) \ldots, \Delta f_n\right)$
R3	R1	$\Rightarrow$	$x_1 = \tau_{\mathrm{f}}\left(f, \sigma_{\mathrm{f}}\left(\Delta f_1\right)\right)$
R4	R3 & 542	$\Rightarrow$	$x_1 = \tau_{\mathrm{f}}\left(f, \Delta f_1\right)$
R5	R2	$\Rightarrow$	$y_1 = \tau_{\mathrm{f}}\left(f, \Delta f_1\right)$
R6	R4 & R5	$\Rightarrow$	$x_1 = y_1$
R7	R2	$\Rightarrow$	$\left(x_k = y_k \Rightarrow y_{k+1} = \tau_{\mathrm{f}}\left(x_k, \Delta f_{k+1}\right)\right)$
R8	437	$\Rightarrow$	$\tau_{\mathrm{f}}\left(x_k, \Delta f_{k+1}\right) = x_k \times \Delta f_{k+1}$
R9	R1 & R8	$\Rightarrow$	$\tau_{\mathrm{f}}\left(x_k, \Delta f_{k+1}\right) = \tau_{\mathrm{f}}\left(f, \sigma_{\mathrm{f}}\left(\Delta f_1, \Delta f_2, \ldots \Delta f_k\right)\right) \times \Delta f_{k+1}$
R10	R9 & 437	$\Rightarrow$	$\tau_{\mathrm{f}}\left(x_k, \Delta f_{k+1}\right) = f \times \sigma_{\mathrm{f}}\left(\Delta f_1, \Delta f_2, \ldots \Delta f_k\right) \times \Delta f_{k+1}$
R11	R10 & 542	$\Rightarrow$	$\tau_{\mathrm{f}}\left(x_k, \Delta f_{k+1}\right) = f \times \prod_{j=1}^{k} \Delta f_j \times \Delta f_{k+1}$
			$= f \times \prod_{j=1}^{k+1} \Delta f_j$
			$= f \times \sigma_{\mathrm{f}}\left(\Delta f_1, \Delta f_2, \ldots \Delta f_{k+1}\right)$
R12	R11 & 437	$\Rightarrow$	$\tau_{\mathrm{f}}\left(x_k, \Delta f_{k+1}\right) = \tau_{\mathrm{f}}\left(f, \sigma_{\mathrm{f}}\left(\Delta f_1, \Delta f_2, \ldots \Delta f_{k+1}\right)\right)$
R13	R1 & R12	$\Rightarrow$	$\tau_{\mathrm{f}}\left(x_k, \Delta f_{k+1}\right) = x_{k+1}$
R14	R7 & R13	$\Rightarrow$	$\left(x_k = y_k \Rightarrow x_{k+1} = y_{k+1}\right)$
R15	R6 & R14	$\Rightarrow$	$x_k = y_k$ for all integer k greater than zero
R16	R1, R2 & R15	$\Rightarrow$	$\tau_{\mathrm{f}}\left(f, \sigma_{\mathrm{f}}\left(\Delta f_1, \Delta f_2, \ldots \Delta f_n\right)\right) = \tau_{\mathrm{f}}\left(\ldots \tau_{\mathrm{f}}\left(\tau_{\mathrm{f}}\left(f, \Delta f_1\right), \Delta f_2\right) \ldots, \Delta f_n\right)$

Inversion of frequency intervals

Definition 544 (Definition of $\iota_{\mathrm{f}}(\Delta f)$) *If ψ is a pitch system and Δf is a frequency interval in ψ and f is a frequency in ψ then $\iota_{\mathrm{f}}(\Delta f)$ is the frequency interval that satisfies the following equation*

$$\tau_{\mathrm{f}}\left(\tau_{\mathrm{f}}(f, \Delta f), \iota_{\mathrm{f}}(\Delta f)\right) = f$$

Definition 545 (Inversional equivalence of frequency intervals) *If ψ is a pitch system and Δf_1 and*

Δf_2 *are frequency intervals in* ψ *then* Δf_1 *and* Δf_2 *are* inversionally equivalent *if and only if*

$$(\iota_{\mathrm{f}}(\Delta f_1) = \Delta f_2) \vee (\Delta f_1 = \Delta f_2)$$

The fact that two frequency intervals are inversionally equivalent is denoted as follows:

$$\Delta f_1 \equiv_{\iota} \Delta f_2$$

Theorem 546 *If*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

is a pitch system and Δf *is a frequency interval in* ψ *then*

$$\iota_{\mathrm{f}}(\Delta f) = \frac{1}{\Delta f}$$

Proof

R1	544	$\Rightarrow$	$\tau_{\mathrm{f}}(\tau_{\mathrm{f}}(f, \Delta f), \iota_{\mathrm{f}}(\Delta f)) = f$
R2	437	$\Rightarrow$	$\tau_{\mathrm{f}}(\tau_{\mathrm{f}}(f, \Delta f), \iota_{\mathrm{f}}(\Delta f))$
			$= \tau_{\mathrm{f}}(f \times \Delta f, \iota_{\mathrm{f}}(\Delta f))$
			$= f \times \Delta f \times \iota_{\mathrm{f}}(\Delta f)$
R3	R1 & R2	$\Rightarrow$	$f \times \Delta f \times \iota_{\mathrm{f}}(\Delta f) = f$
		$\Rightarrow$	$\Delta f \times \iota_{\mathrm{f}}(\Delta f) = 1$
		$\Rightarrow$	$\iota_{\mathrm{f}}(\Delta f) = \frac{1}{\Delta f}$

Theorem 547 *If* ψ *is a pitch system and* Δf*,* Δf_1 *and* Δf_2 *are frequency intervals in* ψ *then*

$$(\Delta f_1 = \iota_{\mathrm{f}}(\Delta f)) \wedge (\Delta f_2 = \iota_{\mathrm{f}}(\Delta f)) \Rightarrow (\Delta f_1 = \Delta f_2)$$

Proof

R1	Let		$\Delta f_1 = \iota_{\mathrm{f}}(\Delta f)$
R2	Let		$\Delta f_2 = \iota_{\mathrm{f}}(\Delta f)$
R3	R1 & 546	$\Rightarrow$	$\Delta f_1 = \frac{1}{\Delta f}$
R4	R2 & 546	$\Rightarrow$	$\Delta f_2 = \frac{1}{\Delta f}$
R5	R3 & R4	$\Rightarrow$	$\Delta f_1 = \Delta f_2$
R6	R1 to R5	$\Rightarrow$	$(\Delta f_1 = \iota_{\mathrm{f}}(\Delta f)) \wedge (\Delta f_2 = \iota_{\mathrm{f}}(\Delta f)) \Rightarrow (\Delta f_1 = \Delta f_2)$

Exponentiation of frequency intervals

Definition 548 (Definition of $\epsilon_{\mathrm{f},n}(\Delta f)$) *Given that:*

1. ψ *is a pitch system;*
2. f *is a frequency in* ψ*;*
3. Δf *is a frequency interval in* ψ*;*
4. n *is an integer;*
5. k *is an integer and* $1 \leq k \leq \mathrm{abs}(n)$*;*
6. $\Delta f_{1,k} = \Delta f$ *for all* k*; and*
7. $\Delta f_{2,k} = \iota_{\mathrm{f}}(\Delta f)$ *for all* k*;*

then $\epsilon_{\mathrm{f},n}(\Delta f)$ *returns a frequency interval that satisfies the following equation:*

$$\tau_{\mathrm{f}}(f, \epsilon_{\mathrm{f},n}(\Delta f)) = \begin{cases} \tau_{\mathrm{f}}(f, \sigma_{\mathrm{f}}(\Delta f_{1,1}, \Delta f_{1,2}, \ldots \Delta f_{1,n})) & \text{if} \quad n > 0 \\ f & \text{if} \quad n = 0 \\ \tau_{\mathrm{f}}(f, \sigma_{\mathrm{f}}(\Delta f_{2,1}, \Delta f_{2,2}, \ldots \Delta f_{2,-n})) & \text{if} \quad n < 0 \end{cases}$$

Theorem 549 (Formula for $\epsilon_{\mathrm{f},n}(\Delta f)$) *If*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

is a pitch system and Δf *is a frequency interval in* ψ *and* n *is an integer then*

$$\epsilon_{\mathrm{f},n}(\Delta f) = (\Delta f)^n$$

Proof

R1	Let		n be an integer
R2	Let		k be an integer such that $1 \leq k \leq \text{abs}(n)$
R3	Let		$\Delta f_{1,k} = \Delta f$ for all k
R4	Let		$\Delta f_{2,k} = \iota_{\mathrm{f}}(\Delta f)$ for all k
R5	Let		n_1 be an integer greater than zero
R6	R2, R3, R5 & 548	$\Rightarrow$	$\tau_{\mathrm{f}}(f, \epsilon_{\mathrm{f},n_1}(\Delta f)) = \tau_{\mathrm{f}}(f, \sigma_{\mathrm{f}}(\Delta f_{1,1}, \Delta f_{1,2}, \ldots \Delta f_{1,n_1}))$
R7	R6 & 542	$\Rightarrow$	$\tau_{\mathrm{f}}(f, \epsilon_{\mathrm{f},n_1}(\Delta f)) = \tau_{\mathrm{f}}\left(f, \prod_{j=1}^{n_1} \Delta f_{1,j}\right)$
R8	R7 & 440	$\Rightarrow$	$\epsilon_{\mathrm{f},n_1}(\Delta f) = \prod_{j=1}^{n_1} \Delta f_{1,j}$
R9	R3 & R8	$\Rightarrow$	$\epsilon_{\mathrm{f},n_1}(\Delta f) = \prod_{j=1}^{n_1} \Delta f = (\Delta f)^{n_1}$
R10	R1, R5 & R9	$\Rightarrow$	$\epsilon_{\mathrm{f},n}(\Delta f) = (\Delta f)^n$ when $n > 0$
R11			$(\Delta f)^0 = 1$
R12	R11	$\Rightarrow$	$\tau_{\mathrm{f}}\left(f, (\Delta f)^0\right) = \tau_{\mathrm{f}}(f, 1)$
R13	R12 & 437	$\Rightarrow$	$\tau_{\mathrm{f}}\left(f, (\Delta f)^0\right) = f \times 1 = f$
R14	548	$\Rightarrow$	$\tau_{\mathrm{f}}(f, \epsilon_{\mathrm{f},0}(\Delta f)) = f$
R15	R13, R14 & 440	$\Rightarrow$	$\epsilon_{\mathrm{f},0}(\Delta f) = (\Delta f)^0$
R16	R1 & R15	$\Rightarrow$	$\epsilon_{\mathrm{f},n}(\Delta f) = (\Delta f)^n$ when $n = 0$
R17	Let		n_2 be any integer less than zero
R18	R4, R17 & 548	$\Rightarrow$	$\tau_{\mathrm{f}}(f, \epsilon_{\mathrm{f},n_2}(\Delta f)) = \tau_{\mathrm{f}}(f, \sigma_{\mathrm{f}}(\Delta f_{2,1}, \Delta f_{2,2}, \ldots \Delta f_{2,-n_2}))$
R19	R18 & 542	$\Rightarrow$	$\tau_{\mathrm{f}}(f, \epsilon_{\mathrm{f},n_2}(\Delta f)) = \tau_{\mathrm{f}}\left(f, \prod_{j=1}^{-n_2} \Delta f_{2,j}\right)$
R20	R19 & 440	$\Rightarrow$	$\epsilon_{\mathrm{f},n_2}(\Delta f) = \prod_{j=1}^{-n_2} \Delta f_{2,j}$
R21	R4 & R20	$\Rightarrow$	$\epsilon_{\mathrm{f},n_2}(\Delta f) = \prod_{j=1}^{-n_2} \iota_{\mathrm{f}}(\Delta f) = (\iota_{\mathrm{f}}(\Delta f))^{-n_2}$
R22	R21 & 546	$\Rightarrow$	$\epsilon_{\mathrm{f},n_2}(\Delta f) = \left(\frac{1}{\Delta f}\right)^{-n_2} = (\Delta f)^{n_2}$
R23	R1, R17 & R22	$\Rightarrow$	$\epsilon_{\mathrm{f},n}(\Delta f) = (\Delta f)^n$ when $n < 0$
R24	R1, R10, R16 & R23	$\Rightarrow$	$\epsilon_{\mathrm{f},n}(\Delta f) = (\Delta f)^n$ for all integer n

Theorem 550 *If*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

is a pitch system and Δf *is any frequency interval in* ψ *then*

$$\iota_{\mathrm{f}}(\Delta f) = \epsilon_{\mathrm{f},-1}(\Delta f)$$

Proof

R1	546	$\Rightarrow$	$\iota_{\mathrm{f}}(\Delta f) = \frac{1}{\Delta f} = (\Delta f)^{-1}$
R2	549	$\Rightarrow$	$\epsilon_{\mathrm{f},-1}(\Delta f) = (\Delta f)^{-1}$
R3	R1 & R2	$\Rightarrow$	$\iota_{\mathrm{f}}(\Delta f) = \epsilon_{\mathrm{f},-1}(\Delta f)$

Theorem 551 *If*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

is a pitch system, $n_1, n_2, \ldots n_k$ *is a collection of integers and* Δf *is a frequency interval in* ψ *then*

$$\epsilon_{\mathrm{f},n_k}(\ldots \epsilon_{\mathrm{f},n_2}(\epsilon_{\mathrm{f},n_1}(\Delta f))\ldots) = \epsilon_{\mathrm{f},\prod_{j=1}^{k} n_j}(\Delta f)$$

Proof

R1	Let		$x_k = \epsilon_{\mathrm{f},n_k}\left(\ldots \epsilon_{\mathrm{f},n_2}\left(\epsilon_{\mathrm{f},n_1}\left(\Delta f\right)\right)\ldots\right)$
R2	Let		$y_k = \epsilon_{\mathrm{f},\prod_{j=1}^{k} n_j}\left(\Delta f\right)$
R3	R1	$\Rightarrow$	$x_1 = \epsilon_{\mathrm{f},n_1}\left(\Delta f\right)$
R4	R2	$\Rightarrow$	$y_1 = \epsilon_{\mathrm{f},\prod_{j=1}^{1} n_j}\left(\Delta f\right) = \epsilon_{\mathrm{f},n_1}\left(\Delta f\right)$
R5	R3 & R4	$\Rightarrow$	$x_1 = y_1$
R6	R1	$\Rightarrow$	$\left(x_k = y_k \Rightarrow x_{k+1} = \epsilon_{\mathrm{f},n_{k+1}}\left(y_k\right)\right)$
R7	R2	$\Rightarrow$	$\epsilon_{\mathrm{f},n_{k+1}}\left(y_k\right) = \epsilon_{\mathrm{f},n_{k+1}}\left(\epsilon_{\mathrm{f},\prod_{j=1}^{k} n_j}\left(\Delta f\right)\right)$
R8	R7 & 549	$\Rightarrow$	$\epsilon_{\mathrm{f},n_{k+1}}\left(y_k\right) = \epsilon_{\mathrm{f},n_{k+1}}\left(\left(\Delta f\right)^{\prod_{j=1}^{k} n_j}\right)$
			$= \left(\left(\Delta f\right)^{\prod_{j=1}^{k} n_j}\right)^{n_{k+1}}$
			$= \left(\Delta f\right)^{n_{k+1} \times \prod_{j=1}^{k} n_j} = \left(\Delta f\right)^{\prod_{j=1}^{k+1} n_j}$
			$= \epsilon_{\mathrm{f},\prod_{j=1}^{k+1} n_j}\left(\Delta f\right)$
R9	R2 & R8	$\Rightarrow$	$\epsilon_{\mathrm{f},n_{k+1}}\left(y_k\right) = y_{k+1}$
R10	R6 & R9	$\Rightarrow$	$\left(x_k = y_k \Rightarrow x_{k+1} = y_{k+1}\right)$
R11	R5 & R10	$\Rightarrow$	$x_k = y_k$ for all integer k greater than zero
R12	R1, R2 & R11	$\Rightarrow$	$\epsilon_{\mathrm{f},n_k}\left(\ldots \epsilon_{\mathrm{f},n_2}\left(\epsilon_{\mathrm{f},n_1}\left(\Delta f\right)\right)\ldots\right) = \epsilon_{\mathrm{f},\prod_{j=1}^{k} n_j}\left(\Delta f\right)$

Theorem 552 *If*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

is a pitch system, n is an integer and Δf is a frequency interval in ψ then

$$\iota_{\mathrm{f}}\left(\epsilon_{\mathrm{f},n}\left(\Delta f\right)\right) = \epsilon_{\mathrm{f},-n}\left(\Delta f\right)$$

Proof

R1	549	$\Rightarrow$	$\iota_{\mathrm{f}}\left(\epsilon_{\mathrm{f},n}\left(\Delta f\right)\right) = \iota_{\mathrm{f}}\left(\left(\Delta f\right)^n\right)$
R2	R1 & 546	$\Rightarrow$	$\iota_{\mathrm{f}}\left(\epsilon_{\mathrm{f},n}\left(\Delta f\right)\right) = \frac{1}{(\Delta f)^n} = \left(\Delta f\right)^{-n}$
R3	R2 & 549	$\Rightarrow$	$\iota_{\mathrm{f}}\left(\epsilon_{\mathrm{f},n}\left(\Delta f\right)\right) = \epsilon_{\mathrm{f},-n}\left(\Delta f\right)$

Theorem 553 *If*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

is a pitch system, n is an integer and Δf is a frequency interval in ψ then:

$$\Delta\,\mathrm{p_c}\left(\epsilon_{\mathrm{f},n}\left(\Delta f\right)\right) = \epsilon_{\mathrm{p_c},n}\left(\Delta\,\mathrm{p_c}\left(\Delta f\right)\right)$$

Proof

R1	549	$\Rightarrow$	$\Delta\,\mathrm{p_c}\left(\epsilon_{\mathrm{f},n}\left(\Delta f\right)\right) = \Delta\,\mathrm{p_c}\left(\left(\Delta f\right)^n\right)$
R2	R1 & 293	$\Rightarrow$	$\Delta\,\mathrm{p_c}\left(\epsilon_{\mathrm{f},n}\left(\Delta f\right)\right) = \mu_{\mathrm{c}} \times \frac{\ln((\Delta f)^n)}{\ln 2}$
			$= n \times \mu_{\mathrm{c}} \times \frac{\ln(\Delta f)}{\ln 2}$
			$= n \times \Delta\,\mathrm{p_c}\left(\Delta f\right)$
R3	R2 & 518	$\Rightarrow$	$\Delta\,\mathrm{p_c}\left(\epsilon_{\mathrm{f},n}\left(\Delta f\right)\right) = \epsilon_{\mathrm{p_c},n}\left(\Delta\,\mathrm{p_c}\left(\Delta f\right)\right)$

Theorem 554 *If*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

is a pitch system, n is an integer and Δf is a frequency interval in ψ then:

$$\Delta\,\mathrm{c}\left(\epsilon_{\mathrm{f},n}\left(\Delta f\right)\right) = \epsilon_{\mathrm{c},n}\left(\Delta\,\mathrm{c}\left(\Delta f\right)\right)$$

Proof

R1	549	$\Rightarrow$	$\Delta\,\mathrm{c}\left(\epsilon_{\mathrm{f},n}\left(\Delta f\right)\right) = \Delta\,\mathrm{c}\left(\left(\Delta f\right)^n\right)$
R2	R1 & 296	$\Rightarrow$	$\Delta\,\mathrm{c}\left(\epsilon_{\mathrm{f},n}\left(\Delta f\right)\right) = \left(\mu_{\mathrm{c}} \times \left(\frac{\ln((\Delta f)^n)}{\ln 2}\right)\right) \bmod \mu_{\mathrm{c}}$
			$= \left(n \times \mu_{\mathrm{c}} \times \frac{\ln(\Delta f)}{\ln 2}\right) \bmod \mu_{\mathrm{c}}$
R3	R2 & 45	$\Rightarrow$	$\Delta\,\mathrm{c}\left(\epsilon_{\mathrm{f},n}\left(\Delta f\right)\right) = \left(n \times \left(\left(\mu_{\mathrm{c}} \times \frac{\ln(\Delta f)}{\ln 2}\right) \bmod \mu_{\mathrm{c}}\right)\right) \bmod \mu_{\mathrm{c}}$
R4	R3 & 296	$\Rightarrow$	$\Delta\,\mathrm{c}\left(\epsilon_{\mathrm{f},n}\left(\Delta f\right)\right) = \left(n \times \Delta\,\mathrm{c}\left(\Delta f\right)\right) \bmod \mu_{\mathrm{c}}$
R5	R4 & 454	$\Rightarrow$	$\Delta\,\mathrm{c}\left(\epsilon_{\mathrm{f},n}\left(\Delta f\right)\right) = \epsilon_{\mathrm{c},n}\left(\Delta\,\mathrm{c}\left(\Delta f\right)\right)$

Theorem 555 *If*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

is a pitch system, $n_1, n_2, \ldots n_k$ is a collection of integers and Δf is a frequency interval in ψ then

$$\sigma_{\mathrm{f}}\left(\epsilon_{\mathrm{f},n_1}\left(\Delta f\right), \epsilon_{\mathrm{f},n_2}\left(\Delta f\right), \ldots, \epsilon_{\mathrm{f},n_k}\left(\Delta f\right)\right) = \epsilon_{\mathrm{f},\sum_{j=1}^{k} n_j}\left(\Delta f\right)$$

Proof

R1	Let		$x_k = \sigma_{\mathrm{f}}\left(\epsilon_{\mathrm{f},n_1}(\Delta f), \epsilon_{\mathrm{f},n_2}(\Delta f), \ldots, \epsilon_{\mathrm{f},n_k}(\Delta f)\right)$
R2	R1 & 542	$\Rightarrow$	$x_k = \prod_{j=1}^{k} \epsilon_{\mathrm{f},n_j}(\Delta f)$
R3	R2 & 549	$\Rightarrow$	$x_k = \prod_{j=1}^{k} (\Delta f)^{n_j}$
			$= (\Delta f)^{\sum_{j=1}^{k} n_j}$
			$= \epsilon_{\mathrm{f},\sum_{j=1}^{k} n_j}(\Delta f)$
R4	R1 & R3	$\Rightarrow$	$\sigma_{\mathrm{f}}\left(\epsilon_{\mathrm{f},n_1}(\Delta f), \epsilon_{\mathrm{f},n_2}(\Delta f), \ldots, \epsilon_{\mathrm{f},n_k}(\Delta f)\right) = \epsilon_{\mathrm{f},\sum_{j=1}^{k} n_j}(\Delta f)$

Exponentiation of the frequency tranposition function

Definition 556 (Definition of $\tau_{\mathrm{f},n}(f, \Delta f)$) *If ψ is a pitch system and f is a frequency in ψ and Δf is a frequency interval in ψ then*

$$\tau_{\mathrm{f},n}(f, \Delta f) = \tau_{\mathrm{f}}\left(f, \epsilon_{\mathrm{f},n}(\Delta f)\right)$$

Theorem 557 *If*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

is a pitch system, $n_1, n_2, \ldots, n_k$ is a collection of integers, f is a frequency in ψ and Δf is a frequency interval in ψ then

$$\tau_{\mathrm{f},n_k}\left(\ldots \tau_{\mathrm{f},n_2}\left(\tau_{\mathrm{f},n_1}(f, \Delta f), \Delta f\right) \ldots, \Delta f\right) = \tau_{\mathrm{f},\sum_{j=1}^{k} n_j}(f, \Delta f)$$

Proof

R1	Let		$x_k = \tau_{\mathrm{f},n_k}\left(\ldots \tau_{\mathrm{f},n_2}\left(\tau_{\mathrm{f},n_1}\left(f,\Delta f\right),\Delta f\right)\ldots,\Delta f\right)$
R2	Let		$y_k = \tau_{\mathrm{f},\sum_{j=1}^{k} n_j}\left(f,\Delta f\right)$
R3	R1	$\Rightarrow$	$x_1 = \tau_{\mathrm{f},n_1}\left(f,\Delta f\right)$
R4	R2	$\Rightarrow$	$y_1 = \tau_{\mathrm{f},\sum_{j=1}^{1} n_j}\left(f,\Delta f\right) = \tau_{\mathrm{f},n_1}\left(f,\Delta f\right)$
R5	R3 & R4	$\Rightarrow$	$x_1 = y_1$
R6	R1	$\Rightarrow$	$\left(x_k = y_k \Rightarrow x_{k+1} = \tau_{\mathrm{f},n_{k+1}}\left(y_k,\Delta f\right)\right)$
R7	R2	$\Rightarrow$	$\tau_{\mathrm{f},n_{k+1}}\left(y_k,\Delta f\right) = \tau_{\mathrm{f},n_{k+1}}\left(\tau_{\mathrm{f},\sum_{j=1}^{k} n_j}\left(f,\Delta f\right),\Delta f\right)$
R8	R7 & 556	$\Rightarrow$	$\tau_{\mathrm{f},n_{k+1}}\left(y_k,\Delta f\right) = \tau_{\mathrm{f},n_{k+1}}\left(\tau_{\mathrm{f}}\left(f,\epsilon_{\mathrm{f},\sum_{j=1}^{k} n_j}\left(\Delta f\right)\right),\Delta f\right)$
			$= \tau_{\mathrm{f}}\left(\tau_{\mathrm{f}}\left(f,\epsilon_{\mathrm{f},\sum_{j=1}^{k} n_j}\left(\Delta f\right)\right),\epsilon_{\mathrm{f},n_{k+1}}\left(\Delta f\right)\right)$
R9	R8 & 549	$\Rightarrow$	$\tau_{\mathrm{f},n_{k+1}}\left(y_k,\Delta f\right) = \tau_{\mathrm{f}}\left(\tau_{\mathrm{f}}\left(f,\left(\Delta f\right)^{\sum_{j=1}^{k} n_j}\right),\left(\Delta f\right)^{n_{k+1}}\right)$
R10	R9 & 437	$\Rightarrow$	$\tau_{\mathrm{f},n_{k+1}}\left(y_k,\Delta f\right) = \tau_{\mathrm{f}}\left(f\times\left(\Delta f\right)^{\sum_{j=1}^{k} n_j},\left(\Delta f\right)^{n_{k+1}}\right)$
			$= f\times\left(\Delta f\right)^{\sum_{j=1}^{k} n_j}\times\left(\Delta f\right)^{n_{k+1}}$
			$= f\times\left(\Delta f\right)^{\sum_{j=1}^{k+1} n_j}$
			$= \tau_{\mathrm{f}}\left(f,\left(\Delta f\right)^{\sum_{j=1}^{k+1} n_j}\right)$
R11	R10 & 549	$\Rightarrow$	$\tau_{\mathrm{f},n_{k+1}}\left(y_k,\Delta f\right) = \tau_{\mathrm{f}}\left(f,\epsilon_{\mathrm{f},\sum_{j=1}^{k+1} n_j}\left(\Delta f\right)\right)$
R12	R11 & 556	$\Rightarrow$	$\tau_{\mathrm{f},n_{k+1}}\left(y_k,\Delta f\right) = \tau_{\mathrm{f},\sum_{j=1}^{k+1} n_j}\left(f,\Delta f\right)$
R13	R12 & R2	$\Rightarrow$	$\tau_{\mathrm{f},n_{k+1}}\left(y_k,\Delta f\right) = y_{k+1}$
R14	R13 & R6	$\Rightarrow$	$\left(x_k = y_k \Rightarrow x_{k+1} = y_{k+1}\right)$
R15	R5 & R14	$\Rightarrow$	$x_k = y_k$ for all integers k greater than zero
R16	R1, R2 & R15	$\Rightarrow$	$\tau_{\mathrm{f},n_k}\left(\ldots \tau_{\mathrm{f},n_2}\left(\tau_{\mathrm{f},n_1}\left(f,\Delta f\right),\Delta f\right)\ldots,\Delta f\right) = \tau_{\mathrm{f},\sum_{j=1}^{k} n_j}\left(f,\Delta f\right)$

4.6.8 Summation, inversion and exponentiation of pitch intervals

Summation of pitch intervals

Definition 558 (Definition of $\sigma_{\mathrm{p}}\left(\Delta p_1, \Delta p_2, \ldots, \Delta p_n\right)$) *If*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

is a pitch system and

$$\Delta p_1, \Delta p_2, \ldots, \Delta p_n$$

is a collection of pitch intervals in ψ then

$$\sigma_{\mathrm{p}}\left(\Delta p_1, \Delta p_2, \ldots, \Delta p_n\right) = \left[\sum_{k=1}^{n}\left(\Delta\,\mathrm{p_c}\left(\Delta p_k\right)\right), \sum_{k=1}^{n}\left(\Delta\,\mathrm{p_m}\left(\Delta p_k\right)\right)\right]$$

Theorem 559 *If*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

is a pitch system and

$$\Delta p_1, \Delta p_2, \ldots, \Delta p_n$$

is a collection of pitch intervals in ψ then

$$\sigma_{\mathrm{p}}\left(\Delta p_1, \Delta p_2, \ldots, \Delta p_n\right) = \left[\begin{array}{l} \sigma_{\mathrm{p_c}}\left(\Delta\,\mathrm{p_c}\left(\Delta p_1\right), \Delta\,\mathrm{p_c}\left(\Delta p_2\right), \ldots \Delta\,\mathrm{p_c}\left(\Delta p_k\right), \ldots \Delta\,\mathrm{p_c}\left(\Delta p_n\right)\right), \\ \\ \sigma_{\mathrm{p_m}}\left(\Delta\,\mathrm{p_m}\left(\Delta p_1\right), \Delta\,\mathrm{p_m}\left(\Delta p_2\right), \ldots \Delta\,\mathrm{p_m}\left(\Delta p_k\right), \ldots \Delta\,\mathrm{p_m}\left(\Delta p_n\right)\right) \end{array}\right]$$

Proof

R1	Let		$x_n = \sigma_{\mathrm{p}}\left(\Delta p_1, \Delta p_2, \ldots, \Delta p_n\right)$
R2	Let		$y_n = \sigma_{\mathrm{p_c}}\left(\Delta\,\mathrm{p_c}\left(\Delta p_1\right), \Delta\,\mathrm{p_c}\left(\Delta p_2\right), \ldots \Delta\,\mathrm{p_c}\left(\Delta p_k\right), \ldots \Delta\,\mathrm{p_c}\left(\Delta p_n\right)\right)$
R3	Let		$\sigma_{\mathrm{p_m}}\left(\Delta\,\mathrm{p_m}\left(\Delta p_1\right), \Delta\,\mathrm{p_m}\left(\Delta p_2\right), \ldots \Delta\,\mathrm{p_m}\left(\Delta p_k\right), \ldots \Delta\,\mathrm{p_m}\left(\Delta p_n\right)\right)$
R4	558 & R1	$\Rightarrow$	$x_n = \left[\sum_{k=1}^{n}\left(\Delta\,\mathrm{p_c}\left(\Delta p_k\right)\right), \sum_{k=1}^{n}\left(\Delta\,\mathrm{p_m}\left(\Delta p_k\right)\right)\right]$
R5	511 & R2	$\Rightarrow$	$y_n = \sum_{k=1}^{n}\left(\Delta\,\mathrm{p_c}\left(\Delta p_k\right)\right)$
R6	527 & R3	$\Rightarrow$	$z_n = \sum_{k=1}^{n}\left(\Delta\,\mathrm{p_m}\left(\Delta p_k\right)\right)$
R7	R4, R5 & R6	$\Rightarrow$	$x_n = [y_n, z_n]$
R8	R7, R1, R2 & R3	$\Rightarrow$	$\sigma_{\mathrm{p}}\left(\Delta p_1, \Delta p_2, \ldots, \Delta p_n\right)$

$$= \left[\begin{array}{l} \sigma_{\mathrm{p_c}}\left(\Delta\,\mathrm{p_c}\left(\Delta p_1\right), \Delta\,\mathrm{p_c}\left(\Delta p_2\right), \ldots \Delta\,\mathrm{p_c}\left(\Delta p_k\right), \ldots \Delta\,\mathrm{p_c}\left(\Delta p_n\right)\right), \\ \\ \sigma_{\mathrm{p_m}}\left(\Delta\,\mathrm{p_m}\left(\Delta p_1\right), \Delta\,\mathrm{p_m}\left(\Delta p_2\right), \ldots \Delta\,\mathrm{p_m}\left(\Delta p_k\right), \ldots \Delta\,\mathrm{p_m}\left(\Delta p_n\right)\right) \end{array}\right]$$

Theorem 560 *If ψ is a pitch system and*

$$\Delta p_1, \Delta p_2, \ldots, \Delta p_n$$

is a collection of pitch intervals in ψ and p is a pitch in ψ then

$$\tau_{\mathrm{p}}\left(p, \sigma_{\mathrm{p}}\left(\Delta p_1, \Delta p_2, \ldots, \Delta p_n\right)\right) = \tau_{\mathrm{p}}\left(\ldots \tau_{\mathrm{p}}\left(\tau_{\mathrm{p}}\left(p, \Delta p_1\right), \Delta p_2\right) \ldots, \Delta p_n\right)$$

Proof

R1	Let		$x_n = \tau_{\mathrm{p}}\left(p, \sigma_{\mathrm{p}}\left(\Delta p_1, \Delta p_2, \ldots, \Delta p_n\right)\right)$
R2	Let		$y_n = \tau_{\mathrm{p}}\left(\ldots \tau_{\mathrm{p}}\left(\tau_{\mathrm{p}}\left(p, \Delta p_1\right), \Delta p_2\right) \ldots, \Delta p_n\right)$
R3	R1	$\Rightarrow$	$x_1 = \tau_{\mathrm{p}}\left(p, \sigma_{\mathrm{p}}\left(\Delta p_1\right)\right)$
R4	R3 & 558	$\Rightarrow$	$x_1 = \tau_{\mathrm{p}}\left(p, \left[\sum_{k=1}^{1}\left(\Delta\, \mathrm{p_c}\left(\Delta p_k\right)\right), \sum_{k=1}^{1}\left(\Delta\, \mathrm{p_m}\left(\Delta p_k\right)\right)\right]\right)$
			$= \tau_{\mathrm{p}}\left(p, \left[\Delta\, \mathrm{p_c}\left(\Delta p_1\right), \Delta\, \mathrm{p_m}\left(\Delta p_1\right)\right]\right)$
R5	R4 & 270	$\Rightarrow$	$x_1 = \tau_{\mathrm{p}}\left(p, \Delta p_1\right)$
R6	R2	$\Rightarrow$	$y_1 = \tau_{\mathrm{p}}\left(p, \Delta p_1\right)$
R7	R5 & R6	$\Rightarrow$	$x_1 = y_1$
R8	R1 & R2	$\Rightarrow$	$\left(x_k = y_k \Rightarrow y_{k+1} = \tau_{\mathrm{p}}\left(x_k, \Delta p_{k+1}\right)\right)$
R9	R1	$\Rightarrow$	$\tau_{\mathrm{p}}\left(x_k, \Delta p_{k+1}\right) = \tau_{\mathrm{p}}\left(\tau_{\mathrm{p}}\left(p, \sigma_{\mathrm{p}}\left(\Delta p_1, \Delta p_2, \ldots, \Delta p_k\right)\right), \Delta p_{k+1}\right)$
R10	R9 & 558	$\Rightarrow$	$\tau_{\mathrm{p}}\left(x_k, \Delta p_{k+1}\right) = \tau_{\mathrm{p}}\left(\tau_{\mathrm{p}}\left(p, \left[\sum_{j=1}^{k}\left(\Delta\, \mathrm{p_c}\left(\Delta p_j\right)\right), \sum_{j=1}^{k}\left(\Delta\, \mathrm{p_m}\left(\Delta p_j\right)\right)\right]\right), \Delta p_{k+1}\right)$
R11	R10, 442, 267 & 269	$\Rightarrow$	$\tau_{\mathrm{p}}\left(x_k, \Delta p_{k+1}\right) = \tau_{\mathrm{p}}\left(\left[\begin{array}{c} \tau_{\mathrm{p_c}}\left(\mathrm{p_c}\left(p\right), \sum_{j=1}^{k}\left(\Delta\, \mathrm{p_c}\left(\Delta p_j\right)\right)\right), \\ \\ \tau_{\mathrm{p_m}}\left(\mathrm{p_m}\left(p\right), \sum_{j=1}^{k}\left(\Delta\, \mathrm{p_m}\left(\Delta p_j\right)\right)\right) \end{array}\right], \Delta p_{k+1}\right)$
R12	R11, 427 & 432	$\Rightarrow$	$\tau_{\mathrm{p}}\left(x_k, \Delta p_{k+1}\right) = \tau_{\mathrm{p}}\left(\left[\begin{array}{c} \mathrm{p_c}\left(p\right) + \sum_{j=1}^{k}\left(\Delta\, \mathrm{p_c}\left(\Delta p_j\right)\right), \\ \\ \mathrm{p_m}\left(p\right) + \sum_{j=1}^{k}\left(\Delta\, \mathrm{p_m}\left(\Delta p_j\right)\right) \end{array}\right], \Delta p_{k+1}\right)$
R13	R12, 442, 63 & 64	$\Rightarrow$	$\tau_{\mathrm{p}}\left(x_k, \Delta p_{k+1}\right) = \left[\begin{array}{c} \tau_{\mathrm{p_c}}\left(\mathrm{p_c}\left(p\right) + \sum_{j=1}^{k}\left(\Delta\, \mathrm{p_c}\left(\Delta p_j\right)\right), \Delta\, \mathrm{p_c}\left(\Delta p_{k+1}\right)\right), \\ \\ \tau_{\mathrm{p_m}}\left(\mathrm{p_m}\left(p\right) + \sum_{j=1}^{k}\left(\Delta\, \mathrm{p_m}\left(\Delta p_j\right)\right), \Delta\, \mathrm{p_m}\left(\Delta p_{k+1}\right)\right) \end{array}\right]$
R14	R13, 427 & 432	$\Rightarrow$	$\tau_{\mathrm{p}}\left(x_k, \Delta p_{k+1}\right) = \left[\begin{array}{c} \mathrm{p_c}\left(p\right) + \sum_{j=1}^{k}\left(\Delta\, \mathrm{p_c}\left(\Delta p_j\right)\right) + \Delta\, \mathrm{p_c}\left(\Delta p_{k+1}\right), \\ \\ \mathrm{p_m}\left(p\right) + \sum_{j=1}^{k}\left(\Delta\, \mathrm{p_m}\left(\Delta p_j\right)\right) + \Delta\, \mathrm{p_m}\left(\Delta p_{k+1}\right) \end{array}\right]$
			$= \left[\mathrm{p_c}\left(p\right) + \sum_{j=1}^{k+1}\left(\Delta\, \mathrm{p_c}\left(\Delta p_j\right)\right), \mathrm{p_m}\left(p\right) + \sum_{j=1}^{k+1}\left(\Delta\, \mathrm{p_m}\left(\Delta p_j\right)\right)\right]$
			$= \left[\tau_{\mathrm{p_c}}\left(\mathrm{p_c}\left(p\right), \sum_{j=1}^{k+1}\left(\Delta\, \mathrm{p_c}\left(\Delta p_j\right)\right)\right), \tau_{\mathrm{p_m}}\left(\mathrm{p_m}\left(p\right), \sum_{j=1}^{k+1}\left(\Delta\, \mathrm{p_m}\left(\Delta p_j\right)\right)\right)\right]$

R15	R14, 442, 267 & 269	$\Rightarrow$	$\tau_{\mathrm{p}}(x_k, \Delta p_{k+1}) = \tau_{\mathrm{p}}\left(p, \left[\sum_{j=1}^{k+1}(\Delta\,\mathrm{p_c}(\Delta p_j)), \sum_{j=1}^{k+1}(\Delta\,\mathrm{p_m}(\Delta p_j))\right]\right)$
R16	R15 & 558	$\Rightarrow$	$\tau_{\mathrm{p}}(x_k, \Delta p_{k+1}) = \tau_{\mathrm{p}}(p, \sigma_{\mathrm{p}}(\Delta p_1, \Delta p_2, \ldots, \Delta p_{k+1}))$
R17	R16 & R1	$\Rightarrow$	$\tau_{\mathrm{p}}(x_k, \Delta p_{k+1}) = x_{k+1}$
R18	R17 & R8	$\Rightarrow$	$(x_k = y_k \Rightarrow x_{k+1} = y_{k+1})$
R19	R18 & R7	$\Rightarrow$	$x_k = y_k$ for all integers k greater than zero
R20	R19, R1 & R2	$\Rightarrow$	$\tau_{\mathrm{p}}(p, \sigma_{\mathrm{p}}(\Delta p_1, \Delta p_2, \ldots, \Delta p_n)) = \tau_{\mathrm{p}}(\ldots\tau_{\mathrm{p}}(\tau_{\mathrm{p}}(p, \Delta p_1), \Delta p_2)\ldots, \Delta p_n)$

Inversion of pitch intervals

Definition 561 (Inverse of a pitch interval) *If ψ is a pitch system and Δp is a pitch interval in ψ and p is a pitch in ψ then the* inverse of Δp, *denoted $\iota_{\mathrm{p}}(\Delta p)$, is the pitch interval that satisfies the following equation*

$$\tau_{\mathrm{p}}(\tau_{\mathrm{p}}(p, \Delta p), \iota_{\mathrm{p}}(\Delta p)) = p$$

Definition 562 (Inversional equivalence of pitch intervals) *If ψ is a pitch system and Δp_1 and Δp_2 are pitch intervals in ψ then Δp_1 and Δp_2 are* inversionally equivalent *if and only if*

$$(\iota_{\mathrm{p}}(\Delta p_1) = \Delta p_2) \vee (\Delta p_1 = \Delta p_2)$$

The fact that two pitch intervals are inversionally equivalent is denoted as follows:

$$\Delta p_1 \equiv_{\iota} \Delta p_2$$

Theorem 563 *If*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

is a pitch system and Δp is a pitch interval in ψ then

$$\iota_{\mathrm{p}}(\Delta p) = [-\Delta\,\mathrm{p_c}(\Delta p), -\Delta\,\mathrm{p_m}(\Delta p)]$$

Proof

R1	561	$\Rightarrow$	$\tau_{\mathrm{p}}\left(\tau_{\mathrm{p}}\left(p,\Delta p\right),\iota_{\mathrm{p}}\left(\Delta p\right)\right)=p$
R2	R1 & 446	$\Rightarrow$	$p=\tau_{\mathrm{p}}\left(\left[\mathrm{p_c}\left(p\right)+\Delta\,\mathrm{p_c}\left(\Delta p\right),\mathrm{p_m}\left(p\right)+\Delta\,\mathrm{p_m}\left(\Delta p\right)\right],\iota_{\mathrm{p}}\left(\Delta p\right)\right)$
R3	R2, 63, 64 & 446	$\Rightarrow$	$p=\left[\mathrm{p_c}\left(p\right)+\Delta\,\mathrm{p_c}\left(\Delta p\right)+\Delta\,\mathrm{p_c}\left(\iota_{\mathrm{p}}\left(\Delta p\right)\right),\mathrm{p_m}\left(p\right)+\Delta\,\mathrm{p_m}\left(\Delta p\right)+\Delta\,\mathrm{p_m}\left(\iota_{\mathrm{p}}\left(\Delta p\right)\right)\right]$
R4	R3 & 63	$\Rightarrow$	$\mathrm{p_c}\left(p\right)=\mathrm{p_c}\left(p\right)+\Delta\,\mathrm{p_c}\left(\Delta p\right)+\Delta\,\mathrm{p_c}\left(\iota_{\mathrm{p}}\left(\Delta p\right)\right)$
		$\Rightarrow$	$\Delta\,\mathrm{p_c}\left(\iota_{\mathrm{p}}\left(\Delta p\right)\right)=-\Delta\,\mathrm{p_c}\left(\Delta p\right)$
R5	R3 & 64	$\Rightarrow$	$\mathrm{p_m}\left(p\right)=\mathrm{p_m}\left(p\right)+\Delta\,\mathrm{p_m}\left(\Delta p\right)+\Delta\,\mathrm{p_m}\left(\iota_{\mathrm{p}}\left(\Delta p\right)\right)$
		$\Rightarrow$	$\Delta\,\mathrm{p_m}\left(\iota_{\mathrm{p}}\left(\Delta p\right)\right)=-\Delta\,\mathrm{p_m}\left(\Delta p\right)$
R6	R4, R5 & 270	$\Rightarrow$	$\iota_{\mathrm{p}}\left(\Delta p\right)=\left[-\Delta\,\mathrm{p_c}\left(\Delta p\right),-\Delta\,\mathrm{p_m}\left(\Delta p\right)\right]$

Theorem 564 *If*

$$\psi=\left[\mu_{\mathrm{c}},\mu_{\mathrm{m}},f_0,p_{\mathrm{c},0}\right]$$

is a pitch system and Δp is a pitch interval in ψ then

$$\iota_{\mathrm{p}}\left(\Delta p\right)=\left[\iota_{\mathrm{p_c}}\left(\Delta\,\mathrm{p_c}\left(\Delta p\right)\right),\iota_{\mathrm{p_m}}\left(\Delta\,\mathrm{p_m}\left(\Delta p\right)\right)\right]$$

Proof

R1	563	$\Rightarrow$	$\iota_{\mathrm{p}}\left(\Delta p\right)=\left[-\Delta\,\mathrm{p_c}\left(\Delta p\right),-\Delta\,\mathrm{p_m}\left(\Delta p\right)\right]$
R2	515	$\Rightarrow$	$-\Delta\,\mathrm{p_c}\left(\Delta p\right)=\iota_{\mathrm{p_c}}\left(\Delta\,\mathrm{p_c}\left(\Delta p\right)\right)$
R3	531	$\Rightarrow$	$-\Delta\,\mathrm{p_m}\left(\Delta p\right)=\iota_{\mathrm{p_m}}\left(\Delta\,\mathrm{p_m}\left(\Delta p\right)\right)$
R4	R1, R2 & R3	$\Rightarrow$	$\iota_{\mathrm{p}}\left(\Delta p\right)=\left[\iota_{\mathrm{p_c}}\left(\Delta\,\mathrm{p_c}\left(\Delta p\right)\right),\iota_{\mathrm{p_m}}\left(\Delta\,\mathrm{p_m}\left(\Delta p\right)\right)\right]$

Theorem 565 *If ψ is a pitch system and Δp, Δp_1 and Δp_2 are pitch intervals in ψ then*

$$\left(\Delta p_1=\iota_{\mathrm{p}}\left(\Delta p\right)\right)\wedge\left(\Delta p_2=\iota_{\mathrm{p}}\left(\Delta p\right)\right)\Rightarrow\left(\Delta p_1=\Delta p_2\right)$$

Proof

R1	Let		$\Delta p_1 = \iota_{\mathrm{p}}(\Delta p)$
R2	Let		$\Delta p_2 = \iota_{\mathrm{p}}(\Delta p)$
R3	R1 & 563	$\Rightarrow$	$\Delta p_1 = [-\Delta\,\mathrm{p_c}(\Delta p), -\Delta\,\mathrm{p_m}(\Delta p)]$
R4	R2 & 563	$\Rightarrow$	$\Delta p_2 = [-\Delta\,\mathrm{p_c}(\Delta p), -\Delta\,\mathrm{p_m}(\Delta p)]$
R5	R3 & R4	$\Rightarrow$	$\Delta p_1 = \Delta p_2$
R6	R1 to R5	$\Rightarrow$	$(\Delta p_1 = \iota_{\mathrm{p}}(\Delta p)) \wedge (\Delta p_2 = \iota_{\mathrm{p}}(\Delta p)) \Rightarrow (\Delta p_1 = \Delta p_2)$

Exponentiation of pitch intervals

Definition 566 (Definition of $\epsilon_{\mathrm{p},n}(\Delta p)$) *Given that:*

1. *ψ is a pitch system;*
2. *p is a pitch in ψ;*
3. *Δp is a pitch interval in ψ;*
4. *n is an integer;*
5. *k is an integer and $1 \leq k \leq \mathrm{abs}(n)$;*
6. *$\Delta p_{1,k} = \Delta p$ for all k; and*
7. *$\Delta p_{2,k} = \iota_{\mathrm{p}}(\Delta p)$ for all k;*

then $\epsilon_{\mathrm{p},n}(\Delta p)$ returns a pitch interval that satisfies the following equation:

$$\tau_{\mathrm{p}}(p, \epsilon_{\mathrm{p},n}(\Delta p)) = \begin{cases} \tau_{\mathrm{p}}(p, \sigma_{\mathrm{p}}(\Delta p_{1,1}, \Delta p_{1,2}, \ldots \Delta p_{1,n})) & \text{if} \quad n > 0 \\ p & \text{if} \quad n = 0 \\ \tau_{\mathrm{p}}(p, \sigma_{\mathrm{p}}(\Delta p_{2,1}, \Delta p_{2,2}, \ldots \Delta p_{2,-n})) & \text{if} \quad n < 0 \end{cases}$$

Theorem 567 (Formula for $\epsilon_{\mathrm{p},n}(\Delta p)$) *If*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

is a pitch system and Δp is a pitch interval in ψ and n is an integer then

$$\epsilon_{\mathrm{p},n}(\Delta p) = [n \times \Delta\,\mathrm{p_c}(\Delta p), n \times \Delta\,\mathrm{p_m}(\Delta p)]$$

Proof

R1	Let		n_1 be any integer greater than zero.
R2	Let		$\Delta p_{1,k} = \Delta p$ for all integer k
R3	Let		$\Delta p_{2,k} = \iota_{\mathrm{p}}(\Delta p)$ for all integer k
R4	566, R2 & R1	$\Rightarrow$	$\tau_{\mathrm{p}}(p, \epsilon_{\mathrm{p},n_1}(\Delta p)) = \tau_{\mathrm{p}}(p, \sigma_{\mathrm{p}}(\Delta p_{1,1}, \Delta p_{1,2}, \ldots \Delta p_{1,n_1}))$
R5	445 & R4	$\Rightarrow$	$\epsilon_{\mathrm{p},n_1}(\Delta p) = \sigma_{\mathrm{p}}(\Delta p_{1,1}, \Delta p_{1,2}, \ldots \Delta p_{1,n_1})$
R6	558 & R5	$\Rightarrow$	$\epsilon_{\mathrm{p},n_1}(\Delta p) = [\sum_{k=1}^{n_1}(\Delta\,\mathrm{p_c}(\Delta p_{1,k})), \sum_{k=1}^{n_1}(\Delta\,\mathrm{p_m}(\Delta p_{1,k}))]$
R7	R2 & R6	$\Rightarrow$	$\epsilon_{\mathrm{p},n_1}(\Delta p) = [\sum_{k=1}^{n_1}(\Delta\,\mathrm{p_c}(\Delta p)), \sum_{k=1}^{n_1}(\Delta\,\mathrm{p_m}(\Delta p))]$
			$= [n_1 \times \Delta\,\mathrm{p_c}(\Delta p), n_1 \times \Delta\,\mathrm{p_m}(\Delta p)]$
R8	R1 & R7	$\Rightarrow$	$\epsilon_{\mathrm{p},n}(\Delta p) = [n \times \Delta\,\mathrm{p_c}(\Delta p), n \times \Delta\,\mathrm{p_m}(\Delta p)]$ for all integers n greater than zero
R9	Let		n_2 be any integer less than zero.
R10	R3, R9 & 566	$\Rightarrow$	$\tau_{\mathrm{p}}(p, \epsilon_{\mathrm{p},n_2}(\Delta p)) = \tau_{\mathrm{p}}(p, \sigma_{\mathrm{p}}(\Delta p_{2,1}, \Delta p_{2,2}, \ldots \Delta p_{2,-n_2}))$
R11	R10 & 445	$\Rightarrow$	$\epsilon_{\mathrm{p},n_2}(\Delta p) = \sigma_{\mathrm{p}}(\Delta p_{2,1}, \Delta p_{2,2}, \ldots \Delta p_{2,-n_2})$
R12	558 & R11	$\Rightarrow$	$\epsilon_{\mathrm{p},n_2}(\Delta p) = \left[\sum_{k=1}^{-n_2}(\Delta\,\mathrm{p_c}(\Delta p_{2,k})), \sum_{k=1}^{-n_2}(\Delta\,\mathrm{p_m}(\Delta p_{2,k}))\right]$
R13	R3 & R12	$\Rightarrow$	$\epsilon_{\mathrm{p},n_2}(\Delta p) = \left[\sum_{k=1}^{-n_2}(\Delta\,\mathrm{p_c}(\iota_{\mathrm{p}}(\Delta p))), \sum_{k=1}^{-n_2}(\Delta\,\mathrm{p_m}(\iota_{\mathrm{p}}(\Delta p)))\right]$
R14	563 & 267	$\Rightarrow$	$\Delta\,\mathrm{p_c}(\iota_{\mathrm{p}}(\Delta p)) = -\Delta\,\mathrm{p_c}(\Delta p)$
R15	563 & 269	$\Rightarrow$	$\Delta\,\mathrm{p_m}(\iota_{\mathrm{p}}(\Delta p)) = -\Delta\,\mathrm{p_m}(\Delta p)$
R16	R13, R14 & R15	$\Rightarrow$	$\epsilon_{\mathrm{p},n_2}(\Delta p) = \left[\sum_{k=1}^{-n_2}(-\Delta\,\mathrm{p_c}(\Delta p)), \sum_{k=1}^{-n_2}(-\Delta\,\mathrm{p_m}(\Delta p))\right]$
			$= [-n_2 \times (-\Delta\,\mathrm{p_c}(\Delta p)), -n_2 \times (-\Delta\,\mathrm{p_m}(\Delta p))]$
			$= [n_2 \times (\Delta\,\mathrm{p_c}(\Delta p)), n_2 \times (\Delta\,\mathrm{p_m}(\Delta p))]$
R17	R9 & R16	$\Rightarrow$	$\epsilon_{\mathrm{p},n}(\Delta p) = [n \times \Delta\,\mathrm{p_c}(\Delta p), n \times \Delta\,\mathrm{p_m}(\Delta p)]$ for all integers n less than zero.
R18	566	$\Rightarrow$	$\tau_{\mathrm{p}}(p, \epsilon_{\mathrm{p},0}(\Delta p)) = p$
R19	446 & R18	$\Rightarrow$	$p = [\mathrm{p_c}(p) + \Delta\,\mathrm{p_c}(\epsilon_{\mathrm{p},0}(\Delta p)), \mathrm{p_m}(p) + \Delta\,\mathrm{p_m}(\epsilon_{\mathrm{p},0}(\Delta p))]$

R20	R19 & 65	$\Rightarrow$	$[\mathrm{p_c}(p), \mathrm{p_m}(p)] = [\mathrm{p_c}(p) + \Delta\,\mathrm{p_c}(\epsilon_{\mathrm{p},0}(\Delta p)), \mathrm{p_m}(p) + \Delta\,\mathrm{p_m}(\epsilon_{\mathrm{p},0}(\Delta p))]$
R21	R20	$\Rightarrow$	$\mathrm{p_c}(p) = \mathrm{p_c}(p) + \Delta\,\mathrm{p_c}(\epsilon_{\mathrm{p},0}(\Delta p)) \Rightarrow \Delta\,\mathrm{p_c}(\epsilon_{\mathrm{p},0}(\Delta p)) = 0$
R22	R20	$\Rightarrow$	$\mathrm{p_m}(p) = \mathrm{p_m}(p) + \Delta\,\mathrm{p_m}(\epsilon_{\mathrm{p},0}(\Delta p)) \Rightarrow \Delta\,\mathrm{p_m}(\epsilon_{\mathrm{p},0}(\Delta p)) = 0$
R23	R21, R22 & 65	$\Rightarrow$	$\epsilon_{\mathrm{p},0}(\Delta p) = [0, 0]$
R24	R23	$\Rightarrow$	$\epsilon_{\mathrm{p},n}(\Delta p) = [n \times \Delta\,\mathrm{p_c}(\Delta p), n \times \Delta\,\mathrm{p_m}(\Delta p)]$ when $n = 0$
R25	R8, R17 & R24	$\Rightarrow$	$\epsilon_{\mathrm{p},n}(\Delta p) = [n \times \Delta\,\mathrm{p_c}(\Delta p), n \times \Delta\,\mathrm{p_m}(\Delta p)]$ for all integers n

Theorem 568 *If*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

is a pitch system and Δp *is any pitch interval in* ψ *then*

$$\iota_{\mathrm{p}}(\Delta p) = \epsilon_{\mathrm{p},-1}(\Delta p)$$

Proof

R1	563	$\Rightarrow$	$\iota_{\mathrm{p}}(\Delta p) = [-\Delta\,\mathrm{p_c}(\Delta p), -\Delta\,\mathrm{p_m}(\Delta p)]$
R2	567	$\Rightarrow$	$\epsilon_{\mathrm{p},-1}(\Delta p) = [-1 \times \Delta\,\mathrm{p_c}(\Delta p), -1 \times \Delta\,\mathrm{p_m}(\Delta p)]$
R3	R1 & R2	$\Rightarrow$	$\iota_{\mathrm{p}}(\Delta p) = \epsilon_{\mathrm{p},-1}(\Delta p)$

Theorem 569 *If*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

is a pitch system, $n_1, n_2, \ldots n_k$ *is a collection of integers and* Δp *is a pitch interval in* ψ *then*

$$\epsilon_{\mathrm{p},n_k}(\ldots \epsilon_{\mathrm{p},n_2}(\epsilon_{\mathrm{p},n_1}(\Delta p))\ldots) = \epsilon_{\mathrm{p},\prod_{j=1}^{k} n_j}(\Delta p)$$

Proof

R1	Let		$x_k = \epsilon_{\mathrm{p},n_k}\left(\ldots\epsilon_{\mathrm{p},n_2}\left(\epsilon_{\mathrm{p},n_1}\left(\Delta p\right)\right)\ldots\right)$
R2	Let		$y_k = \epsilon_{\mathrm{p},\prod_{j=1}^{k} n_j}\left(\Delta p\right)$
R3	R1	$\Rightarrow$	$x_1 = \epsilon_{\mathrm{p},n_1}\left(\Delta p\right)$
R4	R2	$\Rightarrow$	$y_1 = \epsilon_{\mathrm{p},\prod_{j=1}^{1} n_j}\left(\Delta p\right) = \epsilon_{\mathrm{p},n_1}\left(\Delta p\right)$
R5	R3 & R4	$\Rightarrow$	$x_1 = y_1$
R6	R1	$\Rightarrow$	$\left(x_k = y_k \Rightarrow x_{k+1} = \epsilon_{\mathrm{p},n_{k+1}}\left(y_k\right)\right)$
R7	R2	$\Rightarrow$	$\epsilon_{\mathrm{p},n_{k+1}}\left(y_k\right) = \epsilon_{\mathrm{p},n_{k+1}}\left(\epsilon_{\mathrm{p},\prod_{j=1}^{k} n_j}\left(\Delta p\right)\right)$
R8	R7 & 567	$\Rightarrow$	$\epsilon_{\mathrm{p},n_{k+1}}\left(y_k\right) = \epsilon_{\mathrm{p},n_{k+1}}\left(\left[\prod_{j=1}^{k} n_j \times \Delta\,\mathrm{p_c}\left(\Delta p\right), \prod_{j=1}^{k} n_j \times \Delta\,\mathrm{p_m}\left(\Delta p\right)\right]\right)$
R9	R8, 567, 267 & 269	$\Rightarrow$	$\epsilon_{\mathrm{p},n_{k+1}}\left(y_k\right) = \left[n_{k+1} \times \prod_{j=1}^{k} n_j \times \Delta\,\mathrm{p_c}\left(\Delta p\right), n_{k+1} \times \prod_{j=1}^{k} n_j \times \Delta\,\mathrm{p_m}\left(\Delta p\right)\right]$
			$= \left[\prod_{j=1}^{k+1} n_j \times \Delta\,\mathrm{p_c}\left(\Delta p\right), \prod_{j=1}^{k+1} n_j \times \Delta\,\mathrm{p_m}\left(\Delta p\right)\right]$
R10	R9 & 567	$\Rightarrow$	$\epsilon_{\mathrm{p},n_{k+1}}\left(y_k\right) = \epsilon_{\mathrm{p},\prod_{j=1}^{k+1} n_j}\left(\Delta p\right)$
R11	R2 & R10	$\Rightarrow$	$\epsilon_{\mathrm{p},n_{k+1}}\left(y_k\right) = y_{k+1}$
R12	R6 & R11	$\Rightarrow$	$\left(x_k = y_k \Rightarrow x_{k+1} = y_{k+1}\right)$
R13	R5 & R12	$\Rightarrow$	$x_k = y_k$ for all integers k greater than zero.
R14	R1, R2 & R13	$\Rightarrow$	$\epsilon_{\mathrm{p},n_k}\left(\ldots\epsilon_{\mathrm{p},n_2}\left(\epsilon_{\mathrm{p},n_1}\left(\Delta p\right)\right)\ldots\right) = \epsilon_{\mathrm{p},\prod_{j=1}^{k} n_j}\left(\Delta p\right)$

Theorem 570 *If*

$$\psi = [\mu_\mathrm{c}, \mu_\mathrm{m}, f_0, p_{\mathrm{c},0}]$$

is a pitch system, n is an integer and Δp is a pitch interval in ψ then

$$\iota_\mathrm{p}\left(\epsilon_{\mathrm{p},n}\left(\Delta p\right)\right) = \epsilon_{\mathrm{p},-n}\left(\Delta p\right)$$

Proof

R1	568	$\Rightarrow$	$\iota_\mathrm{p}\left(\epsilon_{\mathrm{p},n}\left(\Delta p\right)\right) = \epsilon_{\mathrm{p},-1}\left(\epsilon_{\mathrm{p},n}\left(\Delta p\right)\right)$
R2	569 & R1	$\Rightarrow$	$\iota_\mathrm{p}\left(\epsilon_{\mathrm{p},n}\left(\Delta p\right)\right) = \epsilon_{\mathrm{p},(-1\times n)}\left(\Delta p\right) = \epsilon_{\mathrm{p},-n}\left(\Delta p\right)$

Theorem 571 *If*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

is a pitch system, n is an integer and Δp is a pitch interval in ψ then:

$$\Delta\,\mathrm{c}\left(\epsilon_{\mathrm{p},n}\left(\Delta p\right)\right) = \epsilon_{\mathrm{c},n}\left(\Delta\,\mathrm{c}\left(\Delta p\right)\right)$$

Proof

R1	567	$\Rightarrow$	$\Delta\,\mathrm{c}\left(\epsilon_{\mathrm{p},n}\left(\Delta p\right)\right) = \Delta\,\mathrm{c}\left(\left[n \times \Delta\,\mathrm{p_c}\left(\Delta p\right), n \times \Delta\,\mathrm{p_m}\left(\Delta p\right)\right]\right)$
R2	274, 267 & R1	$\Rightarrow$	$\Delta\,\mathrm{c}\left(\epsilon_{\mathrm{p},n}\left(\Delta p\right)\right) = \left(n \times \Delta\,\mathrm{p_c}\left(\Delta p\right)\right) \bmod \mu_{\mathrm{c}}$
R3	454	$\Rightarrow$	$\epsilon_{\mathrm{c},n}\left(\Delta\,\mathrm{c}\left(\Delta p\right)\right) = \left(n \times \Delta\,\mathrm{c}\left(\Delta p\right)\right) \bmod \mu_{\mathrm{c}}$
R4	274 & R3	$\Rightarrow$	$\epsilon_{\mathrm{c},n}\left(\Delta\,\mathrm{c}\left(\Delta p\right)\right) = \left(n \times \left(\Delta\,\mathrm{p_c}\left(\Delta p\right) \bmod \mu_{\mathrm{c}}\right)\right) \bmod \mu_{\mathrm{c}}$
R5	R4 & 45	$\Rightarrow$	$\epsilon_{\mathrm{c},n}\left(\Delta\,\mathrm{c}\left(\Delta p\right)\right) = \left(n \times \Delta\,\mathrm{p_c}\left(\Delta p\right)\right) \bmod \mu_{\mathrm{c}}$
R6	R2 & R5	$\Rightarrow$	$\Delta\,\mathrm{c}\left(\epsilon_{\mathrm{p},n}\left(\Delta p\right)\right) = \epsilon_{\mathrm{c},n}\left(\Delta\,\mathrm{c}\left(\Delta p\right)\right)$

Theorem 572 *If*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

is a pitch system, n is an integer and Δp is a pitch interval in ψ then:

$$\Delta\,\mathrm{m}\left(\epsilon_{\mathrm{p},n}\left(\Delta p\right)\right) = \epsilon_{\mathrm{m},n}\left(\Delta\,\mathrm{m}\left(\Delta p\right)\right)$$

Proof

R1	567	$\Rightarrow$	$\Delta\,\mathrm{m}\left(\epsilon_{\mathrm{p},n}\left(\Delta p\right)\right) = \Delta\,\mathrm{m}\left(\left[n \times \Delta\,\mathrm{p_c}\left(\Delta p\right), n \times \Delta\,\mathrm{p_m}\left(\Delta p\right)\right]\right)$
R2	276, 269 & R1	$\Rightarrow$	$\Delta\,\mathrm{m}\left(\epsilon_{\mathrm{p},n}\left(\Delta p\right)\right) = \left(n \times \Delta\,\mathrm{p_m}\left(\Delta p\right)\right) \bmod \mu_{\mathrm{m}}$
R3	468	$\Rightarrow$	$\epsilon_{\mathrm{m},n}\left(\Delta\,\mathrm{m}\left(\Delta p\right)\right) = \left(n \times \Delta\,\mathrm{m}\left(\Delta p\right)\right) \bmod \mu_{\mathrm{m}}$
R4	276 & R3	$\Rightarrow$	$\epsilon_{\mathrm{m},n}\left(\Delta\,\mathrm{m}\left(\Delta p\right)\right) = \left(n \times \left(\Delta\,\mathrm{p_m}\left(\Delta p\right) \bmod \mu_{\mathrm{m}}\right)\right) \bmod \mu_{\mathrm{m}}$
R5	R4 & 45	$\Rightarrow$	$\epsilon_{\mathrm{m},n}\left(\Delta\,\mathrm{m}\left(\Delta p\right)\right) = \left(n \times \Delta\,\mathrm{p_m}\left(\Delta p\right)\right) \bmod \mu_{\mathrm{m}}$
R6	R2 & R5	$\Rightarrow$	$\Delta\,\mathrm{m}\left(\epsilon_{\mathrm{p},n}\left(\Delta p\right)\right) = \epsilon_{\mathrm{m},n}\left(\Delta\,\mathrm{m}\left(\Delta p\right)\right)$

Theorem 573 *If*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

is a pitch system, n is an integer and Δp is a pitch interval in ψ then:

$$\Delta\,\mathrm{q}\left(\epsilon_{\mathrm{p},n}\left(\Delta p\right)\right) = \epsilon_{\mathrm{q},n}\left(\Delta\,\mathrm{q}\left(\Delta p\right)\right)$$

Proof

R1	482	$\Rightarrow$	$\epsilon_{\mathrm{q},n}\left(\Delta\,\mathrm{q}\left(\Delta p\right)\right)=\left[\epsilon_{\mathrm{c},n}\left(\Delta\,\mathrm{c}\left(\Delta\,\mathrm{q}\left(\Delta p\right)\right)\right),\epsilon_{\mathrm{m},n}\left(\Delta\,\mathrm{m}\left(\Delta\,\mathrm{q}\left(\Delta p\right)\right)\right)\right]$
R2	301, 304 & R1	$\Rightarrow$	$\epsilon_{\mathrm{q},n}\left(\Delta\,\mathrm{q}\left(\Delta p\right)\right)=\left[\epsilon_{\mathrm{c},n}\left(\Delta\,\mathrm{c}\left(\Delta p\right)\right),\epsilon_{\mathrm{m},n}\left(\Delta\,\mathrm{m}\left(\Delta p\right)\right)\right]$
R3	571, 572 & R2	$\Rightarrow$	$\epsilon_{\mathrm{q},n}\left(\Delta\,\mathrm{q}\left(\Delta p\right)\right)=\left[\Delta\,\mathrm{c}\left(\epsilon_{\mathrm{p},n}\left(\Delta p\right)\right),\Delta\,\mathrm{m}\left(\epsilon_{\mathrm{p},n}\left(\Delta p\right)\right)\right]$
R4	R3, 301 & 304	$\Rightarrow$	$\epsilon_{\mathrm{q},n}\left(\Delta\,\mathrm{q}\left(\Delta p\right)\right)=\left[\Delta\,\mathrm{c}\left(\Delta\,\mathrm{q}\left(\epsilon_{\mathrm{p},n}\left(\Delta p\right)\right)\right),\Delta\,\mathrm{m}\left(\Delta\,\mathrm{q}\left(\epsilon_{\mathrm{p},n}\left(\Delta p\right)\right)\right)\right]$
R5	R4 & 305	$\Rightarrow$	$\Delta\,\mathrm{q}\left(\epsilon_{\mathrm{p},n}\left(\Delta p\right)\right)=\epsilon_{\mathrm{q},n}\left(\Delta\,\mathrm{q}\left(\Delta p\right)\right)$

Theorem 574 *If ψ is a pitch system, n is an integer and Δp is a pitch interval in ψ then:*

$$\Delta\,\mathrm{g}\left(\epsilon_{\mathrm{p},n}\left(\Delta p\right)\right)=\epsilon_{\mathrm{g},n}\left(\Delta\,\mathrm{g}\left(\Delta p\right)\right)$$

Proof

R1	282	$\Rightarrow \Delta\,\mathrm{g}\left(\epsilon_{\mathrm{p},n}\left(\Delta p\right)\right) = \left[\Delta\,\mathrm{g_c}\left(\epsilon_{\mathrm{p},n}\left(\Delta p\right)\right), \Delta\,\mathrm{m}\left(\epsilon_{\mathrm{p},n}\left(\Delta p\right)\right)\right]$
R2	567	$\Rightarrow \Delta\,\mathrm{g_c}\left(\epsilon_{\mathrm{p},n}\left(\Delta p\right)\right) = \Delta\,\mathrm{g_c}\left(\left[n \times \Delta\,\mathrm{p_c}\left(\Delta p\right), n \times \Delta\,\mathrm{p_m}\left(\Delta p\right)\right]\right)$
R3	R2, 280, 267 & 269	$\Rightarrow \Delta\,\mathrm{g_c}\left(\epsilon_{\mathrm{p},n}\left(\Delta p\right)\right) = n \times \Delta\,\mathrm{p_c}\left(\Delta p\right) - \mu_\mathrm{c} \times \left(\left(n \times \Delta\,\mathrm{p_m}\left(\Delta p\right)\right) \mathrm{div}\ \mu_\mathrm{m}\right)$
R4	572	$\Rightarrow \Delta\,\mathrm{m}\left(\epsilon_{\mathrm{p},n}\left(\Delta p\right)\right) = \epsilon_{\mathrm{m},n}\left(\Delta\,\mathrm{m}\left(\Delta p\right)\right)$
R5	R1, R3 & R4	$\Rightarrow \Delta\,\mathrm{g}\left(\epsilon_{\mathrm{p},n}\left(\Delta p\right)\right) = \begin{bmatrix} n \times \Delta\,\mathrm{p_c}\left(\Delta p\right) - \mu_\mathrm{c} \times \left(\left(n \times \Delta\,\mathrm{p_m}\left(\Delta p\right)\right) \mathrm{div}\ \mu_\mathrm{m}\right), \\ \epsilon_{\mathrm{m},n}\left(\Delta\,\mathrm{m}\left(\Delta p\right)\right) \end{bmatrix}$
R6	501	$\Rightarrow \epsilon_{\mathrm{g},n}\left(\Delta\,\mathrm{g}\left(\Delta p\right)\right) = \begin{bmatrix} n \times \Delta\,\mathrm{g_c}\left(\Delta\,\mathrm{g}\left(\Delta p\right)\right) \\ -\mu_\mathrm{c} \times \left(\left(n \times \Delta\,\mathrm{m}\left(\Delta\,\mathrm{g}\left(\Delta p\right)\right)\right) \mathrm{div}\ \mu_\mathrm{m}\right), \\ \left(n \times \Delta\,\mathrm{m}\left(\Delta\,\mathrm{g}\left(\Delta p\right)\right)\right) \mathrm{mod}\ \mu_\mathrm{m} \end{bmatrix}$
R7	317, 311 & R6	$\Rightarrow \epsilon_{\mathrm{g},n}\left(\Delta\,\mathrm{g}\left(\Delta p\right)\right) = \begin{bmatrix} n \times \Delta\,\mathrm{g_c}\left(\Delta p\right) - \mu_\mathrm{c} \times \left(\left(n \times \Delta\,\mathrm{m}\left(\Delta p\right)\right) \mathrm{div}\ \mu_\mathrm{m}\right), \\ \left(n \times \Delta\,\mathrm{m}\left(\Delta p\right)\right) \mathrm{mod}\ \mu_\mathrm{m} \end{bmatrix}$
R8	R7, 280 & 276	$\Rightarrow \epsilon_{\mathrm{g},n}\left(\Delta\,\mathrm{g}\left(\Delta p\right)\right) = \begin{bmatrix} n \times \left(\Delta\,\mathrm{p_c}\left(\Delta p\right) - \mu_\mathrm{c} \times \left(\Delta\,\mathrm{p_m}\left(\Delta p\right) \mathrm{div}\ \mu_\mathrm{m}\right)\right) \\ -\mu_\mathrm{c} \times \left(\left(n \times \left(\Delta\,\mathrm{p_m}\left(\Delta p\right) \mathrm{mod}\ \mu_\mathrm{m}\right)\right) \mathrm{div}\ \mu_\mathrm{m}\right), \\ \left(n \times \Delta\,\mathrm{m}\left(\Delta p\right)\right) \mathrm{mod}\ \mu_\mathrm{m} \end{bmatrix}$
		$\Rightarrow \epsilon_{\mathrm{g},n}\left(\Delta\,\mathrm{g}\left(\Delta p\right)\right) = \begin{bmatrix} n \times \Delta\,\mathrm{p_c}\left(\Delta p\right) \\ -\mu_\mathrm{c} \times \begin{pmatrix} n \times \left(\Delta\,\mathrm{p_m}\left(\Delta p\right) \mathrm{div}\ \mu_\mathrm{m}\right) \\ +\left(\left(n \times \left(\Delta\,\mathrm{p_m}\left(\Delta p\right) \mathrm{mod}\ \mu_\mathrm{m}\right)\right) \mathrm{div}\ \mu_\mathrm{m}\right) \end{pmatrix}, \\ \left(n \times \Delta\,\mathrm{m}\left(\Delta p\right)\right) \mathrm{mod}\ \mu_\mathrm{m} \end{bmatrix}$
R9	R8 & 58	$\Rightarrow \epsilon_{\mathrm{g},n}\left(\Delta\,\mathrm{g}\left(\Delta p\right)\right) = \begin{bmatrix} n \times \Delta\,\mathrm{p_c}\left(\Delta p\right) - \mu_\mathrm{c} \times \left(\left(n \times \Delta\,\mathrm{p_m}\left(\Delta p\right)\right) \mathrm{div}\ \mu_\mathrm{m}\right), \\ \left(n \times \Delta\,\mathrm{m}\left(\Delta p\right)\right) \mathrm{mod}\ \mu_\mathrm{m} \end{bmatrix}$
R10	R9 & 468	$\Rightarrow \epsilon_{\mathrm{g},n}\left(\Delta\,\mathrm{g}\left(\Delta p\right)\right) = \begin{bmatrix} n \times \Delta\,\mathrm{p_c}\left(\Delta p\right) - \mu_\mathrm{c} \times \left(\left(n \times \Delta\,\mathrm{p_m}\left(\Delta p\right)\right) \mathrm{div}\ \mu_\mathrm{m}\right), \\ \epsilon_{\mathrm{m},n}\left(\Delta\,\mathrm{m}\left(\Delta p\right)\right) \end{bmatrix}$
R11	R10 & R5	$\Rightarrow \Delta\,\mathrm{g}\left(\epsilon_{\mathrm{p},n}\left(\Delta p\right)\right) = \epsilon_{\mathrm{g},n}\left(\Delta\,\mathrm{g}\left(\Delta p\right)\right)$

Theorem 575 *If ψ is a pitch system, n is an integer and Δp is a pitch interval in ψ then:*

$$\Delta\,\mathrm{p_c}\left(\epsilon_{\mathrm{p},n}\left(\Delta p\right)\right) = \epsilon_{\mathrm{p_c},n}\left(\Delta\,\mathrm{p_c}\left(\Delta p\right)\right)$$

Proof

R1	518	$\Rightarrow$	$\epsilon_{\mathrm{p_c},n}\left(\Delta\,\mathrm{p_c}\left(\Delta p\right)\right) = n \times \Delta\,\mathrm{p_c}\left(\Delta p\right)$
R2	567	$\Rightarrow$	$\epsilon_{\mathrm{p},n}\left(\Delta p\right) = \left[n \times \Delta\,\mathrm{p_c}\left(\Delta p\right), n \times \Delta\,\mathrm{p_m}\left(\Delta p\right)\right]$
R3	267 & R2	$\Rightarrow$	$\Delta\,\mathrm{p_c}\left(\epsilon_{\mathrm{p},n}\left(\Delta p\right)\right) = n \times \Delta\,\mathrm{p_c}\left(\Delta p\right)$
R4	R1 & R3	$\Rightarrow$	$\Delta\,\mathrm{p_c}\left(\epsilon_{\mathrm{p},n}\left(\Delta p\right)\right) = \epsilon_{\mathrm{p_c},n}\left(\Delta\,\mathrm{p_c}\left(\Delta p\right)\right)$

Theorem 576 *If ψ is a pitch system, n is an integer and Δp is a pitch interval in ψ then:*

$$\Delta\,\mathrm{p_m}\left(\epsilon_{\mathrm{p},n}\left(\Delta p\right)\right) = \epsilon_{\mathrm{p_m},n}\left(\Delta\,\mathrm{p_m}\left(\Delta p\right)\right)$$

Proof

R1	534	$\Rightarrow$	$\epsilon_{\mathrm{p_m},n}\left(\Delta\,\mathrm{p_m}\left(\Delta p\right)\right) = n \times \Delta\,\mathrm{p_m}\left(\Delta p\right)$
R2	567	$\Rightarrow$	$\epsilon_{\mathrm{p},n}\left(\Delta p\right) = \left[n \times \Delta\,\mathrm{p_c}\left(\Delta p\right), n \times \Delta\,\mathrm{p_m}\left(\Delta p\right)\right]$
R3	269 & R2	$\Rightarrow$	$\Delta\,\mathrm{p_m}\left(\epsilon_{\mathrm{p},n}\left(\Delta p\right)\right) = n \times \Delta\,\mathrm{p_m}\left(\Delta p\right)$
R4	R1 & R3	$\Rightarrow$	$\Delta\,\mathrm{p_m}\left(\epsilon_{\mathrm{p},n}\left(\Delta p\right)\right) = \epsilon_{\mathrm{p_m},n}\left(\Delta\,\mathrm{p_m}\left(\Delta p\right)\right)$

Theorem 577 *If ψ is a pitch system, n is an integer and Δp is a pitch interval in ψ then:*

$$\Delta\,\mathrm{f}\left(\epsilon_{\mathrm{p},n}\left(\Delta p\right)\right) = \epsilon_{\mathrm{f},n}\left(\Delta\,\mathrm{f}\left(\Delta p\right)\right)$$

Proof

R1	549	$\Rightarrow$	$\epsilon_{\mathrm{f},n}\left(\Delta\,\mathrm{f}\left(\Delta p\right)\right) = \left(\Delta\,\mathrm{f}\left(\Delta p\right)\right)^n$
R2	567	$\Rightarrow$	$\epsilon_{\mathrm{p},n}\left(\Delta p\right) = \left[n \times \Delta\,\mathrm{p_c}\left(\Delta p\right), n \times \Delta\,\mathrm{p_m}\left(\Delta p\right)\right]$
R3	272	$\Rightarrow$	$\Delta\,\mathrm{f}\left(\epsilon_{\mathrm{p},n}\left(\Delta p\right)\right) = 2^{\left(\Delta\,\mathrm{p_c}\left(\epsilon_{\mathrm{p},n}\left(\Delta p\right)\right)/\mu_\mathrm{c}\right)}$
R4	R2, R3 & 267	$\Rightarrow$	$\Delta\,\mathrm{f}\left(\epsilon_{\mathrm{p},n}\left(\Delta p\right)\right) = 2^{\left(n \times \Delta\,\mathrm{p_c}\left(\Delta p\right)/\mu_\mathrm{c}\right)}$
			$= \left(2^{\left(\Delta\,\mathrm{p_c}\left(\Delta p\right)/\mu_\mathrm{c}\right)}\right)^n$
R5	R4 & 272	$\Rightarrow$	$\Delta\,\mathrm{f}\left(\epsilon_{\mathrm{p},n}\left(\Delta p\right)\right) = \left(\Delta\,\mathrm{f}\left(\Delta p\right)\right)^n$
R6	R1 & R5	$\Rightarrow$	$\Delta\,\mathrm{f}\left(\epsilon_{\mathrm{p},n}\left(\Delta p\right)\right) = \epsilon_{\mathrm{f},n}\left(\Delta\,\mathrm{f}\left(\Delta p\right)\right)$

Theorem 578 *If*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

is a pitch system, $n_1, n_2, \ldots n_k$ is a collection of integers and Δp is a pitch interval in ψ then

$$\sigma_{\mathrm{p}}\left(\epsilon_{\mathrm{p},n_1}(\Delta p), \epsilon_{\mathrm{p},n_2}(\Delta p), \ldots, \epsilon_{\mathrm{p},n_k}(\Delta p)\right) = \epsilon_{\mathrm{p},\sum_{j=1}^{k} n_j}(\Delta p)$$

Proof

R1	Let		$x_k = \sigma_{\mathrm{p}}\left(\epsilon_{\mathrm{p},n_1}(\Delta p), \epsilon_{\mathrm{p},n_2}(\Delta p), \ldots, \epsilon_{\mathrm{p},n_k}(\Delta p)\right)$
R2	R1 & 558	$\Rightarrow$	$x_k = \left[\sum_{j=1}^{k}\left(\Delta\,\mathrm{p_c}\left(\epsilon_{\mathrm{p},n_j}(\Delta p)\right)\right), \sum_{j=1}^{k}\left(\Delta\,\mathrm{p_m}\left(\epsilon_{\mathrm{p},n_j}(\Delta p)\right)\right)\right]$
R3	567	$\Rightarrow$	$\epsilon_{\mathrm{p},n_j}(\Delta p) = [n_j \times \Delta\,\mathrm{p_c}(\Delta p), n_j \times \Delta\,\mathrm{p_m}(\Delta p)]$
R4	R3, 267, 269 & R2	$\Rightarrow$	$x_k = \left[\sum_{j=1}^{k}\left(n_j \times \Delta\,\mathrm{p_c}(\Delta p)\right), \sum_{j=1}^{k}\left(n_j \times \Delta\,\mathrm{p_m}(\Delta p)\right)\right]$
			$= \left[\left(\sum_{j=1}^{k} n_j\right) \times \Delta\,\mathrm{p_c}(\Delta p), \left(\sum_{j=1}^{k} n_j\right) \times \Delta\,\mathrm{p_m}(\Delta p)\right]$
R5	R4 & 567	$\Rightarrow$	$x_k = \epsilon_{\mathrm{p},\sum_{j=1}^{k} n_j}(\Delta p)$
R6	R1 & R5	$\Rightarrow$	$\sigma_{\mathrm{p}}\left(\epsilon_{\mathrm{p},n_1}(\Delta p), \epsilon_{\mathrm{p},n_2}(\Delta p), \ldots, \epsilon_{\mathrm{p},n_k}(\Delta p)\right) = \epsilon_{\mathrm{p},\sum_{j=1}^{k} n_j}(\Delta p)$

Exponentiation of the pitch tranposition function

Definition 579 (Definition of $\tau_{\mathrm{p},n}(p, \Delta p)$) *If ψ is a pitch system and p is a pitch in ψ and Δp is a pitch interval in ψ then*

$$\tau_{\mathrm{p},n}(p, \Delta p) = \tau_{\mathrm{p}}\left(p, \epsilon_{\mathrm{p},n}(\Delta p)\right)$$

Theorem 580 *If*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

is a pitch system, $n_1, n_2, \ldots, n_k$ is a collection of integers, p is a pitch in ψ and Δp is a pitch interval in ψ then

$$\tau_{\mathrm{p},n_k}\left(\ldots \tau_{\mathrm{p},n_2}\left(\tau_{\mathrm{p},n_1}(p, \Delta p), \Delta p\right) \ldots, \Delta p\right) = \tau_{\mathrm{p},\sum_{j=1}^{k} n_j}(p, \Delta p)$$

Proof

R1	Let		$x_k = \tau_{\mathrm{p},n_k}\left(\ldots \tau_{\mathrm{p},n_2}\left(\tau_{\mathrm{p},n_1}\left(p, \Delta p\right), \Delta p\right)\ldots, \Delta p\right)$
R2	R1 & 579	$\Rightarrow$	$x_k = \tau_{\mathrm{p}}\left(\ldots \tau_{\mathrm{p}}\left(\tau_{\mathrm{p}}\left(p, \epsilon_{\mathrm{p},n_1}\left(\Delta p\right)\right), \epsilon_{\mathrm{p},n_2}\left(\Delta p\right)\right)\ldots, \epsilon_{\mathrm{p},n_k}\left(\Delta p\right)\right)$
R3	R2 & 560	$\Rightarrow$	$x_k = \tau_{\mathrm{p}}\left(p, \sigma_{\mathrm{p}}\left(\epsilon_{\mathrm{p},n_1}\left(\Delta p\right), \epsilon_{\mathrm{p},n_2}\left(\Delta p\right), \ldots \epsilon_{\mathrm{p},n_k}\left(\Delta p\right)\right)\right)$
R4	R3 & 578	$\Rightarrow$	$x_k = \tau_{\mathrm{p}}\left(p, \epsilon_{\mathrm{p},\sum_{j=1}^{k} n_j}\left(\Delta p\right)\right)$
R5	R4 & 579	$\Rightarrow$	$x_k = \tau_{\mathrm{p},\sum_{j=1}^{k} n_j}\left(p, \Delta p\right)$
R6	R1 & R5	$\Rightarrow$	$\tau_{\mathrm{p},n_k}\left(\ldots \tau_{\mathrm{p},n_2}\left(\tau_{\mathrm{p},n_1}\left(p, \Delta p\right), \Delta p\right)\ldots, \Delta p\right) = \tau_{\mathrm{p},\sum_{j=1}^{k} n_j}\left(p, \Delta p\right)$

4.7 Sets of *MIPS* objects

4.7.1 Universal sets of *MIPS* objects

Definition 581 *The* universal set of pitches $\underline{p}_{\mathrm{u}}$ *for a specified pitch system* ψ *is the set that contains all and only pitches within* ψ*.*

Theorem 582 *For a specified pitch system* ψ*,* $\underline{p}_{\mathrm{u}}$ *contains all and only those values* $p = [p_{\mathrm{c}}, p_{\mathrm{m}}]$ *such that*

$$(p_{\mathrm{c}} \in \mathbb{Z}) \wedge (p_{\mathrm{m}} \in \mathbb{Z})$$

where $\mathbb{Z}$ *is the universal set of integers.*

Proof

R1	Let		$p = [p_{\mathrm{c}}, p_{\mathrm{m}}]$ be any pitch whatsoever in a pitch system ψ.
R2	R1 & 62	$\Rightarrow$	p_{c} can only take any integer value.
R3	R1 & 62	$\Rightarrow$	p_{m} can only take any integer value.
R4	R2, R3 & 581	$\Rightarrow$	$\underline{p}_{\mathrm{u}}$ contains all and only those values $p = [p_{\mathrm{c}}, p_{\mathrm{m}}]$
			such that $(p_{\mathrm{c}} \in \mathbb{Z}) \wedge (p_{\mathrm{m}} \in \mathbb{Z})$
			where $\mathbb{Z}$ is the universal set of integers.

Definition 583 *The* universal set of chromatic pitches $\underline{p}_{\mathrm{c,u}}$ *for a specified pitch system* ψ *is the set that contains all and only chromatic pitches within* ψ*.*

Theorem 584 *For a specified pitch system* ψ*,*

$$\underline{p}_{\mathrm{c,u}} = \mathbb{Z}$$

where $\mathbb{Z}$ *is the universal set of integers.*

Proof

R1	Let		$p = [p_\mathrm{c}, p_\mathrm{m}]$ be any pitch whatsoever in a pitch system ψ.
R2	R1 & 62	$\Rightarrow$	p_c can only take any integer value.
R3	R2 & 583	$\Rightarrow$	$\underline{p}_{\mathrm{c,u}} = \mathbb{Z}$ where $\mathbb{Z}$ is the universal set of integers.

Definition 585 *The* universal set of morphetic pitches $\underline{p}_{\mathrm{m,u}}$ *for a specified pitch system* ψ *is the set that contains all and only morphetic pitches within* ψ*.*

Theorem 586 *For a specified pitch system* ψ*,*

$$\underline{p}_{\mathrm{m,u}} = \mathbb{Z}$$

where $\mathbb{Z}$ *is the universal set of integers.*

Proof

R1	Let		$p = [p_\mathrm{c}, p_\mathrm{m}]$ be any pitch whatsoever in a pitch system ψ.
R2	R1 & 62	$\Rightarrow$	p_m can only take any integer value.
R3	R2 & 585	$\Rightarrow$	$\underline{p}_{\mathrm{m,u}} = \mathbb{Z}$ where $\mathbb{Z}$ is the universal set of integers.

Definition 587 *The* universal set of frequencies $\underline{f}_\mathrm{u}$ *for a specified pitch system* ψ *is the set that contains all and only those values that can be taken by the frequency of a pitch within* ψ*.*

Theorem 588 *For a specified pitch system* ψ*,*

$$\underline{f}_\mathrm{u} = \mathbb{R}^+$$

where $\mathbb{R}^+$ *is the universal set of real numbers greater than zero.*

Proof

R1	Let		f be any frequency in ψ.
R2	67 & R1	$\Rightarrow$	f can only take any value such that $f \in \mathbb{R}^+$.
R3	R2 & 587	$\Rightarrow$	$\underline{f}_\mathrm{u} = \mathbb{R}^+$ where $\mathbb{R}^+$ is the universal set of positive real numbers.

Definition 589 *The* universal set of chromae $\underline{c}_\mathrm{u}$ *for a specified pitch system* ψ *is the set that contains all and only those values that can be taken by a chroma in* ψ*.*

Theorem 590 *For a specified pitch system*

$$\psi = [\mu_\mathrm{c}, \mu_\mathrm{m}, f_0, p_{\mathrm{c},0}]$$

$\underline{c}_\mathrm{u}$ *contains all and only those values* c *such that*

$$(c \in \mathbb{Z}) \wedge (0 \leq c < \mu_\mathrm{c})$$

Proof

R1	Let		p be any pitch in ψ.
R2	72 & R1	$\Rightarrow$	$\mathrm{c}(p)$ can only take any value such that $(\mathrm{c}(p) \in \mathbb{Z}) \wedge (0 \leq \mathrm{c}(p) < \mu_{\mathrm{c}})$.
R3	589 & R2	$\Rightarrow$	$\underline{c}_{\mathrm{u}}$ contains all and only those values c such that $(c \in \mathbb{Z}) \wedge (0 \leq c < \mu_{\mathrm{c}})$.

Definition 591 *The* universal set of morphs $\underline{m}_{\mathrm{u}}$ *for a specified pitch system* ψ *is the set that contains all and only those values that can be taken by a morph in* ψ*.*

Theorem 592 *For a specified pitch system*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

$\underline{m}_{\mathrm{u}}$ *contains all and only those values* m *such that*

$$(m \in \mathbb{Z}) \wedge (0 \leq m < \mu_{\mathrm{m}})$$

Proof

R1	Let		p be any pitch in ψ.
R2	77 & R1	$\Rightarrow$	$\mathrm{m}(p)$ can only take any value such that $(\mathrm{m}(p) \in \mathbb{Z}) \wedge (0 \leq \mathrm{m}(p) < \mu_{\mathrm{m}})$.
R3	591 & R2	$\Rightarrow$	$\underline{m}_{\mathrm{u}}$ contains all and only those values m such that $(m \in \mathbb{Z}) \wedge (0 \leq m < \mu_{\mathrm{m}})$.

Definition 593 *The* universal set of chromamorphs $\underline{q}_{\mathrm{u}}$ *for a specified pitch system* ψ *is the set that contains all and only those values that can be taken by a chromamorph in* ψ*.*

Theorem 594 *For a specified pitch system*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

$\underline{q}_{\mathrm{u}}$ *contains all and only those values* $q = [c, m]$ *such that*

$$(c \in \underline{c}_{\mathrm{u}}) \wedge (m \in \underline{m}_{\mathrm{u}})$$

Proof

R1	Let		p be any pitch in ψ.
R2	80 & R1	$\Rightarrow$	$\mathrm{q}(p) = [\mathrm{c}(p), \mathrm{m}(p)]$
R3	Let		$c = \mathrm{c}(p)$
R4	Let		$m = \mathrm{m}(p)$
R5	Let		$q = \mathrm{q}(p)$
R6	R2, R3, R4 & R5	$\Rightarrow$	$q = [c, m]$
R7	R3 & 589	$\Rightarrow$	c can only take any value such that $c \in \underline{c}_{\mathrm{u}}$.
R8	R4 & 591	$\Rightarrow$	m can only take any value such that $m \in \underline{m}_{\mathrm{u}}$.
R9	593, R6, R7 & R8	$\Rightarrow$	$\underline{q}_{\mathrm{u}}$ contains all and only those values $q = [c, m]$ such that $(c \in \underline{c}_{\mathrm{u}}) \wedge (m \in \underline{m}_{\mathrm{u}})$.

Definition 595 *The* universal set of chromatic genera $\underline{g}_{\mathrm{c,u}}$ *for a specified pitch system ψ is the set that contains all and only those values that can be taken by a chromatic genus in ψ.*

Theorem 596 *For a specified pitch system ψ,*

$$\underline{g}_{\mathrm{c,u}} = \mathbb{Z}$$

where $\mathbb{Z}$ is the universal set of integers.

Proof

R1	Let		p be any pitch in ψ.
R2	83	$\Rightarrow$	$\mathrm{g_c}(p)$ can only take any integer value.
R3	R2 & 595	$\Rightarrow$	$\underline{g}_{\mathrm{c,u}} = \mathbb{Z}$

Definition 597 *The* universal set of genera $\underline{g}_{\mathrm{u}}$ *for a specified pitch system ψ is the set that contains all and only those values that can be taken by a genus in ψ.*

Theorem 598 *For a specified pitch system ψ, $\underline{g}_{\mathrm{u}}$ contains all and only those values $g = [g_{\mathrm{c}}, m]$ such that*

$$\left(g_{\mathrm{c}} \in \underline{g}_{\mathrm{c,u}}\right) \wedge \left(m \in \underline{m}_{\mathrm{u}}\right)$$

Proof

R1	Let		p be any pitch in ψ.
R2	84	$\Rightarrow$	$\mathrm{g}(p) = [\mathrm{g_c}(p), \mathrm{m}(p)]$
R3	Let		$\mathrm{g_c}(p) = g_\mathrm{c}$
R4	Let		$\mathrm{m}(p) = m$
R5	Let		$\mathrm{g}(p) = g$
R6	R2 to R5	$\Rightarrow$	$g = [g_\mathrm{c}, m]$
R7	595 & R3	$\Rightarrow$	g_c can only take any value in $\underline{g}_\mathrm{c,u}$.
R8	591 & R4	$\Rightarrow$	m can only take any value in $\underline{m}_\mathrm{u}$.
R9	R6, R7 & R8	$\Rightarrow$	g can only take any value such that $\left(g_\mathrm{c} \in \underline{g}_\mathrm{c,u}\right) \wedge (m \in \underline{m}_\mathrm{u})$.
R10	597, R6 & R9	$\Rightarrow$	$\underline{g}_\mathrm{u}$ contains all and only those values $g = [g_\mathrm{c}, m]$ such that $\left(g_\mathrm{c} \in \underline{g}_\mathrm{c,u}\right) \wedge (m \in \underline{m}_\mathrm{u})$.

4.7.2 Definitions for sets of *MIPS* objects

Definition 599 *If $\underline{p}_\mathrm{u}$ is the universal set of pitches for the pitch system ψ, then $\underline{p}$ is a well-formed* pitch set *in ψ if and only if*

$$\underline{p} \subseteq \underline{p}_\mathrm{u}$$

Definition 600 *If $\underline{p}_\mathrm{c,u}$ is the universal set of chromatic pitches for the pitch system ψ, then $\underline{p}_\mathrm{c}$ is a well-formed* chromatic pitch set *in ψ if and only if*

$$\underline{p}_\mathrm{c} \subseteq \underline{p}_\mathrm{c,u}$$

Definition 601 *If $\underline{p}_\mathrm{m,u}$ is the universal set of morphetic pitches for the pitch system ψ, then $\underline{p}_\mathrm{m}$ is a well-formed* morphetic pitch set *in ψ if and only if*

$$\underline{p}_\mathrm{m} \subseteq \underline{p}_\mathrm{m,u}$$

Definition 602 *If $\underline{f}_\mathrm{u}$ is the universal set of frequencies for the pitch system ψ, then $\underline{f}$ is a well-formed* frequency set *in ψ if and only if*

$$\underline{f} \subseteq \underline{f}_\mathrm{u}$$

Definition 603 *If $\underline{c}_\mathrm{u}$ is the universal set of chromae for the pitch system ψ, then $\underline{c}$ is a well-formed* chroma set *in ψ if and only if*

$$\underline{c} \subseteq \underline{c}_\mathrm{u}$$

Definition 604 *If $\underline{m}_\mathrm{u}$ is the universal set of morphs for the pitch system ψ, then $\underline{m}$ is a well-formed* morph set *in ψ if and only if*

$$\underline{m} \subseteq \underline{m}_\mathrm{u}$$

Definition 605 *If* $\underline{q}_{\mathrm{u}}$ *is the universal set of chromamorphs for the pitch system* ψ*, then* $\underline{q}$ *is a well-formed* chromamorph set *in* ψ *if and only if*

$$\underline{q} \subseteq \underline{q}_{\mathrm{u}}$$

Definition 606 *If* $\underline{g}_{\mathrm{c,u}}$ *is the universal set of chromatic genera for the pitch system* ψ*, then* $\underline{g}_{\mathrm{c}}$ *is a well-formed* chromatic genus set *in* ψ *if and only if*

$$\underline{g}_{\mathrm{c}} \subseteq \underline{g}_{\mathrm{c,u}}$$

Definition 607 *If* $\underline{g}_{\mathrm{u}}$ *is the universal set of genera for the pitch system* ψ*, then* $\underline{g}$ *is a well-formed* genus set *in* ψ *if and only if*

$$\underline{g} \subseteq \underline{g}_{\mathrm{u}}$$

4.7.3 Chroma set number and morph set number

Definition 608 *If* $\underline{c}$ *is any chroma set in a pitch system* ψ*,*

$$\underline{c} = \left\{c_1, c_2, \ldots c_k, \ldots c_{|\underline{c}|}\right\}$$

then the set number *of* $\underline{c}$*,* $\mathrm{n}\left(\underline{c}\right)$ *is given by the following equation:*

$$\mathrm{n}\left(\underline{c}\right) = \sum_{k=1}^{|\underline{c}|} 2^{c_k}$$

Definition 609 *If* $\underline{m}$ *is any morph set in a pitch system* ψ*,*

$$\underline{m} = \left\{m_1, m_2, \ldots m_k, \ldots m_{|\underline{m}|}\right\}$$

then the set number *of* $\underline{m}$*,* $\mathrm{n}\left(\underline{m}\right)$ *is given by the following equation:*

$$\mathrm{n}\left(\underline{m}\right) = \sum_{k=1}^{|\underline{m}|} 2^{m_k}$$

4.7.4 Functions that convert between *MIPS* object sets of different types

Functions that take a *MIPS* pitch set as argument

Definition 610 *If*

$$\underline{p} = \{p_1, p_2, \ldots p_k, \ldots\}$$

is a pitch set in a pitch system ψ*, then the following function returns the chromatic pitch set of* $\underline{p}$*:*

$$\underline{\mathrm{p}}_{\mathrm{c}}\left(\underline{p}\right) = \bigcup_{k=1}^{|\underline{p}|} \{\mathrm{p_c}\left(p_k\right)\}$$

Definition 611 *If*

$$\underline{p} = \{p_1, p_2, \ldots p_k, \ldots\}$$

is a pitch set in a pitch system ψ*, then the following function returns the morphetic pitch set of* $\underline{p}$*:*

$$\underline{\mathrm{p}}_{\mathrm{m}}\left(\underline{p}\right) = \bigcup_{k=1}^{|\underline{p}|} \{\mathrm{p_m}\left(p_k\right)\}$$

Definition 612 *If*

$$\underline{p} = \{p_1, p_2, \ldots p_k, \ldots\}$$

is a pitch set in a pitch system ψ, then the following function returns the frequency set of $\underline{p}$:

$$\underline{\mathrm{f}}\left(\underline{p}\right) = \bigcup_{k=1}^{|\underline{p}|} \{\mathrm{f}\left(p_k\right)\}$$

Definition 613 *If*

$$\underline{p} = \{p_1, p_2, \ldots p_k, \ldots\}$$

is a pitch set in a pitch system ψ, then the following function returns the chroma set of $\underline{p}$:

$$\underline{\mathrm{c}}\left(\underline{p}\right) = \bigcup_{k=1}^{|\underline{p}|} \{\mathrm{c}\left(p_k\right)\}$$

Definition 614 *If*

$$\underline{p} = \{p_1, p_2, \ldots p_k, \ldots\}$$

is a pitch set in a pitch system ψ, then the following function returns the morph set of $\underline{p}$:

$$\underline{\mathrm{m}}\left(\underline{p}\right) = \bigcup_{k=1}^{|\underline{p}|} \{\mathrm{m}\left(p_k\right)\}$$

Definition 615 *If*

$$\underline{p} = \{p_1, p_2, \ldots p_k, \ldots\}$$

is a pitch set in a pitch system ψ, then the following function returns the chromamorph set of $\underline{p}$:

$$\underline{\mathrm{q}}\left(\underline{p}\right) = \bigcup_{k=1}^{|\underline{p}|} \{\mathrm{q}\left(p_k\right)\}$$

Definition 616 *If*

$$\underline{p} = \{p_1, p_2, \ldots p_k, \ldots\}$$

is a pitch set in a pitch system ψ, then the following function returns the chromatic genus set of $\underline{p}$:

$$\underline{\mathrm{g}}_{\mathrm{c}}\left(\underline{p}\right) = \bigcup_{k=1}^{|\underline{p}|} \{\mathrm{g}_{\mathrm{c}}\left(p_k\right)\}$$

Definition 617 *If*

$$\underline{p} = \{p_1, p_2, \ldots p_k, \ldots\}$$

is a pitch set in a pitch system ψ, then the following function returns the genus set of $\underline{p}$:

$$\underline{\mathrm{g}}\left(\underline{p}\right) = \bigcup_{k=1}^{|\underline{p}|} \{\mathrm{g}\left(p_k\right)\}$$

Functions that take a *MIPS* chromatic pitch set as argument

Definition 618 *If*

$$\underline{p}_{\mathrm{c}} = \{p_{\mathrm{c},1}, p_{\mathrm{c},2}, \ldots p_{\mathrm{c},k}, \ldots\}$$

is a chromatic pitch set in a pitch system ψ, then the following function returns the chroma set of $\underline{p}_{\mathrm{c}}$:

$$\underline{\mathrm{c}}\left(\underline{p}_{\mathrm{c}}\right) = \bigcup_{k=1}^{|\underline{p}_{\mathrm{c}}|} \{\mathrm{c}\,(p_{\mathrm{c},k})\}$$

Definition 619 *If*

$$\underline{p}_{\mathrm{c}} = \{p_{\mathrm{c},1}, p_{\mathrm{c},2}, \ldots p_{\mathrm{c},k}, \ldots\}$$

is a chromatic pitch set in a pitch system ψ, then the following function returns the frequency set of $\underline{p}_{\mathrm{c}}$:

$$\underline{\mathrm{f}}\left(\underline{p}_{\mathrm{c}}\right) = \bigcup_{k=1}^{|\underline{p}_{\mathrm{c}}|} \{\mathrm{f}\,(p_{\mathrm{c},k})\}$$

Functions that take a *MIPS* morphetic pitch set as argument

Definition 620 *If*

$$\underline{p}_{\mathrm{m}} = \{p_{\mathrm{m},1}, p_{\mathrm{m},2}, \ldots p_{\mathrm{m},k}, \ldots\}$$

is a morphetic pitch set in a pitch system ψ, then the following function returns the morph set of $\underline{p}_{\mathrm{m}}$:

$$\underline{\mathrm{m}}\left(\underline{p}_{\mathrm{m}}\right) = \bigcup_{k=1}^{|\underline{p}_{\mathrm{m}}|} \{\mathrm{m}\,(p_{\mathrm{m},k})\}$$

Functions that take a *MIPS* frequency set as argument

Definition 621 *If*

$$\underline{f} = \{f_1, f_2, \ldots f_k, \ldots\}$$

is a frequency set in a pitch system ψ, then the following function returns the chromatic pitch set of $\underline{f}$:

$$\underline{\mathrm{p}}_{\mathrm{c}}\left(\underline{f}\right) = \bigcup_{k=1}^{|\underline{f}|} \{\mathrm{p}_{\mathrm{c}}\,(f_k)\}$$

Definition 622 *If*

$$\underline{f} = \{f_1, f_2, \ldots f_k, \ldots\}$$

is a frequency set in a pitch system ψ, then the following function returns the chroma set of $\underline{f}$:

$$\underline{\mathrm{c}}\left(\underline{f}\right) = \bigcup_{k=1}^{|\underline{f}|} \{\mathrm{c}\,(f_k)\}$$

Functions that take a *MIPS* chromamorph set as argument

Definition 623 *If*

$$\underline{q} = \{q_1, q_2, \ldots q_k, \ldots q_n\}$$

is a chromamorph set in a pitch system ψ, then the following function returns the chroma set of $\underline{q}$:

$$\underline{\mathrm{c}}\left(\underline{q}\right) = \bigcup_{k=1}^{|\underline{q}|} \{\mathrm{c}\left(q_k\right)\}$$

Definition 624 *If*

$$\underline{q} = \{q_1, q_2, \ldots q_k, \ldots q_n\}$$

is a chromamorph set in a pitch system ψ, then the following function returns the morph set of $\underline{q}$:

$$\underline{\mathrm{m}}\left(\underline{q}\right) = \bigcup_{k=1}^{|\underline{q}|} \{\mathrm{m}\left(q_k\right)\}$$

Functions that take a *MIPS* chromatic genus set as argument

Definition 625 *If*

$$\underline{g}_{\mathrm{c}} = \{g_{\mathrm{c},1}, g_{\mathrm{c},2}, \ldots g_{\mathrm{c},k}, \ldots\}$$

is a chromatic genus set in a pitch system ψ, then the following function returns the chroma set of $\underline{g}_{\mathrm{c}}$:

$$\underline{\mathrm{c}}\left(\underline{g}_{\mathrm{c}}\right) = \bigcup_{k=1}^{|\underline{g}_{\mathrm{c}}|} \{\mathrm{c}\left(g_{\mathrm{c},k}\right)\}$$

Functions that take a *MIPS* genus set as argument

Definition 626 *If*

$$\underline{g} = \{g_1, g_2, \ldots g_k, \ldots\}$$

is a genus set in a pitch system ψ, then the following function returns the chromatic genus set of $\underline{g}$:

$$\underline{\mathrm{g}}_{\mathrm{c}}\left(\underline{g}\right) = \bigcup_{k=1}^{|\underline{g}|} \{\mathrm{g}_{\mathrm{c}}\left(g_k\right)\}$$

Definition 627 *If*

$$\underline{g} = \{g_1, g_2, \ldots g_k, \ldots\}$$

is a genus set in a pitch system ψ, then the following function returns the morph set of $\underline{g}$:

$$\underline{\mathrm{m}}\left(\underline{g}\right) = \bigcup_{k=1}^{|\underline{g}|} \{\mathrm{m}\left(g_k\right)\}$$

Definition 628 *If*

$$\underline{g} = \{g_1, g_2, \ldots g_k, \ldots\}$$

is a genus set in a pitch system ψ, then the following function returns the chroma set of $\underline{g}$:

$$\underline{\mathrm{c}}\left(\underline{g}\right) = \bigcup_{k=1}^{|\underline{g}|} \{\mathrm{c}\left(g_k\right)\}$$

Definition 629 *If*

$$\underline{g} = \{g_1, g_2, \ldots g_k, \ldots\}$$

is a genus set in a pitch system ψ, then the following function returns the chromamorph set of $\underline{g}$:

$$\underline{\mathrm{q}}\left(\underline{g}\right) = \bigcup_{k=1}^{|\underline{g}|} \{\mathrm{q}\left(g_k\right)\}$$

4.7.5 Equivalence relations between *MIPS* object sets

Equivalence relations between pitch sets

Definition 630 ($\underline{p}_1 \equiv_{\mathrm{p_c}} \underline{p}_2$) *Two pitch sets $\underline{p}_1$ and $\underline{p}_2$ in a well-formed pitch system are* chromatic pitch equivalent *if and only if*

$$\underline{\mathrm{p}}_{\mathrm{c}}\left(\underline{p}_1\right) = \underline{\mathrm{p}}_{\mathrm{c}}\left(\underline{p}_2\right)$$

The fact that two pitch sets are chromatic pitch equivalent will be denoted

$$\underline{p}_1 \equiv_{\mathrm{p_c}} \underline{p}_2$$

Definition 631 ($\underline{p}_1 \equiv_{\mathrm{p_m}} \underline{p}_2$) *Two pitch sets $\underline{p}_1$ and $\underline{p}_2$ in a well-formed pitch system are* morphetic pitch equivalent *if and only if*

$$\underline{\mathrm{p}}_{\mathrm{m}}\left(\underline{p}_1\right) = \underline{\mathrm{p}}_{\mathrm{m}}\left(\underline{p}_2\right)$$

The fact that two pitch sets are morphetic pitch equivalent will be denoted

$$\underline{p}_1 \equiv_{\mathrm{p_m}} \underline{p}_2$$

Definition 632 ($\underline{p}_1 \equiv_{\mathrm{f}} \underline{p}_2$) *Two pitch sets $\underline{p}_1$ and $\underline{p}_2$ in a well-formed pitch system are* frequency equivalent *if and only if*

$$\underline{\mathrm{f}}\left(\underline{p}_1\right) = \underline{\mathrm{f}}\left(\underline{p}_2\right)$$

The fact that two pitch sets are frequency equivalent will be denoted

$$\underline{p}_1 \equiv_{\mathrm{f}} \underline{p}_2$$

Definition 633 ($\underline{p}_1 \equiv_{\mathrm{c}} \underline{p}_2$) *Two pitch sets $\underline{p}_1$ and $\underline{p}_2$ in a well-formed pitch system are* chroma equivalent *if and only if*

$$\underline{\mathrm{c}}\left(\underline{p}_1\right) = \underline{\mathrm{c}}\left(\underline{p}_2\right)$$

The fact that two pitch sets are chroma equivalent will be denoted

$$\underline{p}_1 \equiv_{\mathrm{c}} \underline{p}_2$$

Definition 634 ($\underline{p}_1 \equiv_{\mathrm{m}} \underline{p}_2$) *Two pitch sets $\underline{p}_1$ and $\underline{p}_2$ in a well-formed pitch system are* morph equivalent *if and only if*

$$\underline{\mathrm{m}}\left(\underline{p}_1\right) = \underline{\mathrm{m}}\left(\underline{p}_2\right)$$

The fact that two pitch sets are morph equivalent will be denoted

$$\underline{p}_1 \equiv_{\mathrm{m}} \underline{p}_2$$

Definition 635 ($\underline{p}_1 \equiv_{\mathrm{q}} \underline{p}_2$) *Two pitch sets* $\underline{p}_1$ *and* $\underline{p}_2$ *in a well-formed pitch system are* chromamorph equivalent *if and only if*

$$\underline{\mathrm{q}}\left(\underline{p}_1\right) = \underline{\mathrm{q}}\left(\underline{p}_2\right)$$

The fact that two pitch sets are chromamorph equivalent will be denoted

$$\underline{p}_1 \equiv_{\mathrm{q}} \underline{p}_2$$

Definition 636 ($\underline{p}_1 \equiv_{\mathrm{g_c}} \underline{p}_2$) *Two pitch sets* $\underline{p}_1$ *and* $\underline{p}_2$ *in a well-formed pitch system are* chromatic genus equivalent *if and only if*

$$\underline{\mathrm{g}}_{\mathrm{c}}\left(\underline{p}_1\right) = \underline{\mathrm{g}}_{\mathrm{c}}\left(\underline{p}_2\right)$$

The fact that two pitch sets are chromatic genus equivalent will be denoted

$$\underline{p}_1 \equiv_{\mathrm{g_c}} \underline{p}_2$$

Definition 637 ($\underline{p}_1 \equiv_{\mathrm{g}} \underline{p}_2$) *Two pitch sets* $\underline{p}_1$ *and* $\underline{p}_2$ *in a well-formed pitch system are* genus equivalent *if and only if*

$$\underline{\mathrm{g}}\left(\underline{p}_1\right) = \underline{\mathrm{g}}\left(\underline{p}_2\right)$$

The fact that two pitch sets are genus equivalent will be denoted

$$\underline{p}_1 \equiv_{\mathrm{g}} \underline{p}_2$$

Equivalence relations between chromatic pitch sets

Definition 638 ($\underline{p}_{\mathrm{c},1} \equiv_{\mathrm{f}} \underline{p}_{\mathrm{c},2}$) *Two chromatic pitch sets* $\underline{p}_{\mathrm{c},1}$ *and* $\underline{p}_{\mathrm{c},2}$ *in a well-formed pitch system are* frequency equivalent *if and only if*

$$\underline{\mathrm{f}}\left(\underline{p}_{\mathrm{c},1}\right) = \underline{\mathrm{f}}\left(\underline{p}_{\mathrm{c},2}\right)$$

The fact that two chromatic pitch sets are frequency equivalent will be denoted

$$\underline{p}_{\mathrm{c},1} \equiv_{\mathrm{f}} \underline{p}_{\mathrm{c},2}$$

Definition 639 ($\underline{p}_{\mathrm{c},1} \equiv_{\mathrm{c}} \underline{p}_{\mathrm{c},2}$) *Two chromatic pitch sets* $\underline{p}_{\mathrm{c},1}$ *and* $\underline{p}_{\mathrm{c},2}$ *in a well-formed pitch system are* chroma equivalent *if and only if*

$$\underline{\mathrm{c}}\left(\underline{p}_{\mathrm{c},1}\right) = \underline{\mathrm{c}}\left(\underline{p}_{\mathrm{c},2}\right)$$

The fact that two chromatic pitch sets are chroma equivalent will be denoted

$$\underline{p}_{\mathrm{c},1} \equiv_{\mathrm{c}} \underline{p}_{\mathrm{c},2}$$

Equivalence relations between morphetic pitch sets

Definition 640 ($\underline{p}_{\mathrm{m},1} \equiv_{\mathrm{m}} \underline{p}_{\mathrm{m},2}$) *Two morphetic pitch sets* $\underline{p}_{\mathrm{m},1}$ *and* $\underline{p}_{\mathrm{m},2}$ *in a well-formed pitch system are* morph equivalent *if and only if*

$$\underline{\mathrm{m}}\left(\underline{p}_{\mathrm{m},1}\right) = \underline{\mathrm{m}}\left(\underline{p}_{\mathrm{m},2}\right)$$

The fact that two morphetic pitch sets are morph equivalent will be denoted

$$\underline{p}_{\mathrm{m},1} \equiv_{\mathrm{m}} \underline{p}_{\mathrm{m},2}$$

Equivalence relations between frequency sets

Definition 641 **($\underline{f}_1 \equiv_{\mathrm{p_c}} \underline{f}_2$)** *Two frequency sets $\underline{f}_1$ and $\underline{f}_2$ in a well-formed pitch system are* chromatic pitch equivalent *if and only if*

$$\underline{\mathrm{p}}_{\mathrm{c}}\left(\underline{f}_1\right) = \underline{\mathrm{p}}_{\mathrm{c}}\left(\underline{f}_2\right)$$

The fact that two frequency sets are chromatic pitch equivalent will be denoted

$$\underline{f}_1 \equiv_{\mathrm{p_c}} \underline{f}_2$$

Definition 642 **($\underline{f}_1 \equiv_{\mathrm{c}} \underline{f}_2$)** *Two frequency sets $\underline{f}_1$ and $\underline{f}_2$ in a well-formed pitch system are* chroma equivalent *if and only if*

$$\underline{\mathrm{c}}\left(\underline{f}_1\right) = \underline{\mathrm{c}}\left(\underline{f}_2\right)$$

The fact that two frequency sets are chroma equivalent will be denoted

$$\underline{f}_1 \equiv_{\mathrm{c}} \underline{f}_2$$

Equivalence relations between chromamorph sets

Definition 643 **($\underline{q}_1 \equiv_{\mathrm{c}} \underline{q}_2$)** *Two chromamorph sets $\underline{q}_1$ and $\underline{q}_2$ in a well-formed pitch system are* chroma equivalent *if and only if*

$$\underline{\mathrm{c}}\left(\underline{q}_1\right) = \underline{\mathrm{c}}\left(\underline{q}_2\right)$$

The fact that two chromamorph sets are chroma equivalent will be denoted

$$\underline{q}_1 \equiv_{\mathrm{c}} \underline{q}_2$$

Definition 644 **($\underline{q}_1 \equiv_{\mathrm{m}} \underline{q}_2$)** *Two chromamorph sets $\underline{q}_1$ and $\underline{q}_2$ in a well-formed pitch system are* morph equivalent *if and only if*

$$\underline{\mathrm{m}}\left(\underline{q}_1\right) = \underline{\mathrm{m}}\left(\underline{q}_2\right)$$

The fact that two chromamorph sets are morph equivalent will be denoted

$$\underline{q}_1 \equiv_{\mathrm{m}} \underline{q}_2$$

Equivalence relations between chromatic genus sets

Definition 645 **($\underline{g}_{\mathrm{c},1} \equiv_{\mathrm{c}} \underline{g}_{\mathrm{c},2}$)** *Two chromatic genus sets $\underline{g}_{\mathrm{c},1}$ and $\underline{g}_{\mathrm{c},2}$ in a well-formed pitch system are* chroma equivalent *if and only if*

$$\underline{\mathrm{c}}\left(\underline{g}_{\mathrm{c},1}\right) = \underline{\mathrm{c}}\left(\underline{g}_{\mathrm{c},2}\right)$$

The fact that two chromatic genus sets are chroma equivalent will be denoted

$$\underline{g}_{\mathrm{c},1} \equiv_{\mathrm{c}} \underline{g}_{\mathrm{c},2}$$

Equivalence relations between genus sets

Definition 646 **($\underline{g}_1 \equiv_{\mathrm{g_c}} \underline{g}_2$)** *Two genus sets $\underline{g}_1$ and $\underline{g}_2$ in a well-formed pitch system are* chromatic genus equivalent *if and only if*

$$\underline{\mathrm{g}}_{\mathrm{c}}\left(\underline{g}_1\right) = \underline{\mathrm{g}}_{\mathrm{c}}\left(\underline{g}_2\right)$$

The fact that two genus sets are chromatic genus equivalent will be denoted

$$\underline{g}_1 \equiv_{\mathrm{g_c}} \underline{g}_2$$

Definition 647 ($\underline{g}_1 \equiv_{\mathrm{m}} \underline{g}_2$) *Two genus sets* $\underline{g}_1$ *and* $\underline{g}_2$ *in a well-formed pitch system are* morph equivalent *if and only if*

$$\underline{\mathrm{m}}\left(\underline{g}_1\right) = \underline{\mathrm{m}}\left(\underline{g}_2\right)$$

The fact that two genus sets are morph equivalent will be denoted

$$\underline{g}_1 \equiv_{\mathrm{m}} \underline{g}_2$$

Definition 648 ($\underline{g}_1 \equiv_{\mathrm{c}} \underline{g}_2$) *Two genus sets* $\underline{g}_1$ *and* $\underline{g}_2$ *in a well-formed pitch system are* chroma equivalent *if and only if*

$$\underline{\mathrm{c}}\left(\underline{g}_1\right) = \underline{\mathrm{c}}\left(\underline{g}_2\right)$$

The fact that two genus sets are chroma equivalent will be denoted

$$\underline{g}_1 \equiv_{\mathrm{c}} \underline{g}_2$$

Definition 649 ($\underline{g}_1 \equiv_{\mathrm{q}} \underline{g}_2$) *Two genus sets* $\underline{g}_1$ *and* $\underline{g}_2$ *in a well-formed pitch system are* chromamorph equivalent *if and only if*

$$\underline{\mathrm{q}}\left(\underline{g}_1\right) = \underline{\mathrm{q}}\left(\underline{g}_2\right)$$

The fact that two genus sets are chromamorph equivalent will be denoted

$$\underline{g}_1 \equiv_{\mathrm{q}} \underline{g}_2$$

4.7.6 Sorting *MIPS* object sets

Sorting pitch sets

Definition 650 *If*

$$\underline{p}_1 = \left\{p_{1,1}, p_{1,2}, \ldots, p_{1,k}, \ldots, p_{1,|\underline{p}_1|}\right\}$$

is a pitch set in a well-formed pitch system then the function $\underline{\mathrm{p}} \uparrow_{\mathrm{p_c}} \left(\underline{p}_1\right)$ *returns the unique ordered pitch set*

$$\underline{\mathrm{p}} \uparrow_{\mathrm{p_c}} \left(\underline{p}_1\right) = \left[p_{2,1}, p_{2,2}, \ldots, p_{2,k}, \ldots, p_{2,|\underline{p}_1|}\right]$$

that satisfies the following conditions:

1. $\left(p \in \underline{\mathrm{p}} \uparrow_{\mathrm{p_c}} \left(\underline{p}_1\right)\right) \iff \left(p \in \underline{p}_1\right)$;
2. $\left|\underline{\mathrm{p}} \uparrow_{\mathrm{p_c}} \left(\underline{p}_1\right)\right| = \left|\underline{p}_1\right|$;
3. *For all natural numbers k such that $1 \leq k < |\underline{p}_1|$, it is true that*
$$p_{2,k} \leq_{\mathrm{p_c}} p_{2,k+1}$$
4. *For all natural numbers k such that $1 \leq k < |\underline{p}_1|$, it is true that*
$$(p_{2,k} \equiv_{\mathrm{p_c}} p_{2,k+1}) \Rightarrow (p_{2,k} <_{\mathrm{p_m}} p_{2,k+1})$$

Definition 651 *If*

$$\underline{p}_1 = \left\{p_{1,1}, p_{1,2}, \ldots, p_{1,k}, \ldots, p_{1,|\underline{p}_1|}\right\}$$

is a pitch set in a well-formed pitch system then the function $\underline{\mathrm{p}} \downarrow_{\mathrm{p_c}} \left(\underline{p}_1\right)$ *returns the unique ordered pitch set*

$$\underline{\mathrm{p}} \downarrow_{\mathrm{p_c}} \left(\underline{p}_1\right) = \left[p_{2,1}, p_{2,2}, \ldots, p_{2,k}, \ldots, p_{2,|\underline{p}_1|}\right]$$

that satisfies the following conditions:

1. $\left(p \in \underline{\mathrm{p}} \downarrow_{\mathrm{p_c}} \left(\underline{p}_1\right)\right) \iff \left(p \in \underline{p}_1\right)$;
2. $\left|\underline{\mathrm{p}} \downarrow_{\mathrm{p_c}} \left(\underline{p}_1\right)\right| = \left|\underline{p}_1\right|$;
3. *For all natural numbers k such that $1 \leq k < |\underline{p}_1|$, it is true that*
$$p_{2,k} \geq_{\mathrm{p_c}} p_{2,k+1}$$
4. *For all natural numbers k such that $1 \leq k < |\underline{p}_1|$, it is true that*
$$(p_{2,k} \equiv_{\mathrm{p_c}} p_{2,k+1}) \Rightarrow (p_{2,k} >_{\mathrm{p_m}} p_{2,k+1})$$

Definition 652 *If*
$$\underline{p}_1 = \left\{p_{1,1}, p_{1,2}, \ldots, p_{1,k}, \ldots, p_{1,|\underline{p}_1|}\right\}$$
is a pitch set in a well-formed pitch system then the function $\underline{\mathrm{p}} \uparrow_{\mathrm{p_m}} \left(\underline{p}_1\right)$ returns the unique ordered pitch set
$$\underline{\mathrm{p}} \uparrow_{\mathrm{p_m}} \left(\underline{p}_1\right) = \left[p_{2,1}, p_{2,2}, \ldots, p_{2,k}, \ldots, p_{2,|\underline{p}_1|}\right]$$
that satisfies the following conditions:

1. $\left(p \in \underline{\mathrm{p}} \uparrow_{\mathrm{p_m}} \left(\underline{p}_1\right)\right) \iff \left(p \in \underline{p}_1\right)$;
2. $\left|\underline{\mathrm{p}} \uparrow_{\mathrm{p_m}} \left(\underline{p}_1\right)\right| = \left|\underline{p}_1\right|$;
3. *For all natural numbers k such that $1 \leq k < |\underline{p}_1|$, it is true that*
$$p_{2,k} \leq_{\mathrm{p_m}} p_{2,k+1}$$
4. *For all natural numbers k such that $1 \leq k < |\underline{p}_1|$, it is true that*
$$(p_{2,k} \equiv_{\mathrm{p_m}} p_{2,k+1}) \Rightarrow (p_{2,k} <_{\mathrm{p_c}} p_{2,k+1})$$

Definition 653 *If*
$$\underline{p}_1 = \left\{p_{1,1}, p_{1,2}, \ldots, p_{1,k}, \ldots, p_{1,|\underline{p}_1|}\right\}$$
is a pitch set in a well-formed pitch system then the function $\underline{\mathrm{p}} \downarrow_{\mathrm{p_m}} \left(\underline{p}_1\right)$ returns the unique ordered pitch set
$$\underline{\mathrm{p}} \downarrow_{\mathrm{p_m}} \left(\underline{p}_1\right) = \left[p_{2,1}, p_{2,2}, \ldots, p_{2,k}, \ldots, p_{2,|\underline{p}_1|}\right]$$
that satisfies the following conditions:

1. $\left(p \in \underline{\mathrm{p}} \downarrow_{\mathrm{p_m}} \left(\underline{p}_1\right)\right) \iff \left(p \in \underline{p}_1\right)$;
2. $\left|\underline{\mathrm{p}} \downarrow_{\mathrm{p_m}} \left(\underline{p}_1\right)\right| = \left|\underline{p}_1\right|$;
3. *For all natural numbers k such that $1 \leq k < |\underline{p}_1|$, it is true that*
$$p_{2,k} \geq_{\mathrm{p_m}} p_{2,k+1}$$
4. *For all natural numbers k such that $1 \leq k < |\underline{p}_1|$, it is true that*
$$(p_{2,k} \equiv_{\mathrm{p_m}} p_{2,k+1}) \Rightarrow (p_{2,k} >_{\mathrm{p_c}} p_{2,k+1})$$

Sorting chromatic pitch sets

Definition 654 *If*

$$\underline{p}_{\mathrm{c},1} = \left\{ p_{\mathrm{c},1,1}, p_{\mathrm{c},1,2}, \ldots, p_{\mathrm{c},1,k}, \ldots, p_{\mathrm{c},1,|\underline{p}_{\mathrm{c},1}|} \right\}$$

is a chromatic pitch set in a well-formed pitch system then the function $\underline{\mathrm{p}}_{\mathrm{c}} \uparrow \left(\underline{p}_{\mathrm{c},1}\right)$ *returns the unique ordered chromatic pitch set*

$$\underline{\mathrm{p}}_{\mathrm{c}} \uparrow \left(\underline{p}_{\mathrm{c},1}\right) = \left[p_{\mathrm{c},2,1}, p_{\mathrm{c},2,2}, \ldots, p_{\mathrm{c},2,k}, \ldots, p_{\mathrm{c},2,|\underline{p}_{\mathrm{c},1}|} \right]$$

that satisfies the following conditions:

1. $\left(p_{\mathrm{c}} \in \underline{\mathrm{p}}_{\mathrm{c}} \uparrow \left(\underline{p}_{\mathrm{c},1}\right)\right) \iff \left(p_{\mathrm{c}} \in \underline{p}_{\mathrm{c},1}\right)$;
2. $\left|\underline{\mathrm{p}}_{\mathrm{c}} \uparrow \left(\underline{p}_{\mathrm{c},1}\right)\right| = \left|\underline{p}_{\mathrm{c},1}\right|$;
3. *For all natural numbers k such that $1 \leq k < |\underline{p}_{\mathrm{c},1}|$, it is true that*

$$p_{\mathrm{c},2,k} < p_{\mathrm{c},2,k+1}$$

Definition 655 *If*

$$\underline{p}_{\mathrm{c},1} = \left\{ p_{\mathrm{c},1,1}, p_{\mathrm{c},1,2}, \ldots, p_{\mathrm{c},1,k}, \ldots, p_{\mathrm{c},1,|\underline{p}_{\mathrm{c},1}|} \right\}$$

is a chromatic pitch set in a well-formed pitch system then the function $\underline{\mathrm{p}}_{\mathrm{c}} \downarrow \left(\underline{p}_{\mathrm{c},1}\right)$ *returns the unique ordered chromatic pitch set*

$$\underline{\mathrm{p}}_{\mathrm{c}} \downarrow \left(\underline{p}_{\mathrm{c},1}\right) = \left[p_{\mathrm{c},2,1}, p_{\mathrm{c},2,2}, \ldots, p_{\mathrm{c},2,k}, \ldots, p_{\mathrm{c},2,|\underline{p}_{\mathrm{c},1}|} \right]$$

that satisfies the following conditions:

1. $\left(p_{\mathrm{c}} \in \underline{\mathrm{p}}_{\mathrm{c}} \downarrow \left(\underline{p}_{\mathrm{c},1}\right)\right) \iff \left(p_{\mathrm{c}} \in \underline{p}_{\mathrm{c},1}\right)$;
2. $\left|\underline{\mathrm{p}}_{\mathrm{c}} \downarrow \left(\underline{p}_{\mathrm{c},1}\right)\right| = \left|\underline{p}_{\mathrm{c},1}\right|$;
3. *For all natural numbers k such that $1 \leq k < |\underline{p}_{\mathrm{c},1}|$, it is true that*

$$p_{\mathrm{c},2,k} > p_{\mathrm{c},2,k+1}$$

Sorting morphetic pitch sets

Definition 656 *If*

$$\underline{p}_{\mathrm{m},1} = \left\{ p_{\mathrm{m},1,1}, p_{\mathrm{m},1,2}, \ldots, p_{\mathrm{m},1,k}, \ldots, p_{\mathrm{m},1,|\underline{p}_{\mathrm{m},1}|} \right\}$$

is a morphetic pitch set in a well-formed pitch system then the function $\underline{\mathrm{p}}_{\mathrm{m}} \uparrow \left(\underline{p}_{\mathrm{m},1}\right)$ *returns the unique ordered morphetic pitch set*

$$\underline{\mathrm{p}}_{\mathrm{m}} \uparrow \left(\underline{p}_{\mathrm{m},1}\right) = \left[p_{\mathrm{m},2,1}, p_{\mathrm{m},2,2}, \ldots, p_{\mathrm{m},2,k}, \ldots, p_{\mathrm{m},2,|\underline{p}_{\mathrm{m},1}|} \right]$$

that satisfies the following conditions:

1. $\left(p_{\mathrm{m}} \in \underline{\mathrm{p}}_{\mathrm{m}} \uparrow \left(\underline{p}_{\mathrm{m},1}\right)\right) \iff \left(p_{\mathrm{m}} \in \underline{p}_{\mathrm{m},1}\right)$;
2. $\left|\underline{\mathrm{p}}_{\mathrm{m}} \uparrow \left(\underline{p}_{\mathrm{m},1}\right)\right| = \left|\underline{p}_{\mathrm{m},1}\right|$;
3. *For all natural numbers k such that $1 \leq k < |\underline{p}_{\mathrm{m},1}|$, it is true that*

$$p_{\mathrm{m},2,k} < p_{\mathrm{m},2,k+1}$$

Definition 657 *If*

$$\underline{p}_{\mathrm{m},1} = \left\{ p_{\mathrm{m},1,1}, p_{\mathrm{m},1,2}, \ldots, p_{\mathrm{m},1,k}, \ldots, p_{\mathrm{m},1,|\underline{p}_{\mathrm{m},1}|} \right\}$$

is a morphetic pitch set in a well-formed pitch system then the function $\underline{\mathrm{p}}_{\mathrm{m}} \downarrow \left(\underline{p}_{\mathrm{m},1}\right)$ *returns the unique ordered morphetic pitch set*

$$\underline{\mathrm{p}}_{\mathrm{m}} \downarrow \left(\underline{p}_{\mathrm{m},1}\right) = \left[p_{\mathrm{m},2,1}, p_{\mathrm{m},2,2}, \ldots, p_{\mathrm{m},2,k}, \ldots, p_{\mathrm{m},2,|\underline{p}_{\mathrm{m},1}|} \right]$$

that satisfies the following conditions:

1. $\left(p_{\mathrm{m}} \in \underline{\mathrm{p}}_{\mathrm{m}} \downarrow \left(\underline{p}_{\mathrm{m},1}\right)\right) \iff \left(p_{\mathrm{m}} \in \underline{p}_{\mathrm{m},1}\right)$;
2. $\left|\underline{\mathrm{p}}_{\mathrm{m}} \downarrow \left(\underline{p}_{\mathrm{m},1}\right)\right| = \left|\underline{p}_{\mathrm{m},1}\right|$;
3. *For all natural numbers k such that $1 \leq k < |\underline{p}_{\mathrm{m},1}|$, it is true that*

$$p_{\mathrm{m},2,k} > p_{\mathrm{m},2,k+1}$$

Sorting frequency sets

Definition 658 *If*

$$\underline{f}_1 = \left\{ f_{1,1}, f_{1,2}, \ldots, f_{1,k}, \ldots, f_{1,|\underline{f}_1|} \right\}$$

is a frequency set in a well-formed pitch system then the function $\underline{\mathrm{f}} \uparrow \left(\underline{f}_1\right)$ *returns the unique ordered frequency set*

$$\underline{\mathrm{f}} \uparrow \left(\underline{f}_1\right) = \left[f_{2,1}, f_{2,2}, \ldots, f_{2,k}, \ldots, f_{2,|\underline{f}_1|} \right]$$

that satisfies the following conditions:

1. $\left(f \in \underline{\mathrm{f}} \uparrow \left(\underline{f}_1\right)\right) \iff \left(f \in \underline{f}_1\right)$;
2. $\left|\underline{\mathrm{f}} \uparrow \left(\underline{f}_1\right)\right| = \left|\underline{f}_1\right|$;
3. *For all natural numbers k such that $1 \leq k < |\underline{f}_1|$, it is true that*

$$f_{2,k} < f_{2,k+1}$$

Definition 659 *If*

$$\underline{f}_1 = \left\{ f_{1,1}, f_{1,2}, \ldots, f_{1,k}, \ldots, f_{1,|\underline{f}_1|} \right\}$$

is a frequency set in a well-formed pitch system then the function $\underline{\mathrm{f}} \downarrow \left(\underline{f}_1\right)$ *returns the unique ordered frequency set*

$$\underline{\mathrm{f}} \downarrow \left(\underline{f}_1\right) = \left[f_{2,1}, f_{2,2}, \ldots, f_{2,k}, \ldots, f_{2,|\underline{f}_1|} \right]$$

that satisfies the following conditions:

1. $\left(f \in \underline{\mathrm{f}} \downarrow \left(\underline{f}_1\right)\right) \iff \left(f \in \underline{f}_1\right)$;
2. $\left|\underline{\mathrm{f}} \downarrow \left(\underline{f}_1\right)\right| = \left|\underline{f}_1\right|$;
3. *For all natural numbers k such that $1 \leq k < |\underline{f}_1|$, it is true that*

$$f_{2,k} > f_{2,k+1}$$

Sorting chroma sets

Definition 660 *If*

$$\underline{c}_1 = \left\{c_{1,1}, c_{1,2}, \ldots, c_{1,k}, \ldots, c_{1,|\underline{c}_1|}\right\}$$

is a chroma set in a well-formed pitch system then the function $\underline{\mathrm{c}} \uparrow (\underline{c}_1)$ *returns the unique ordered chroma set*

$$\underline{\mathrm{c}} \uparrow (\underline{c}_1) = \left[c_{2,1}, c_{2,2}, \ldots, c_{2,k}, \ldots, c_{2,|\underline{c}_1|}\right]$$

that satisfies the following conditions:

1. $(c \in \underline{\mathrm{c}} \uparrow (\underline{c}_1)) \iff (c \in \underline{c}_1)$;
2. $|\underline{\mathrm{c}} \uparrow (\underline{c}_1)| = |\underline{c}_1|$;
3. *For all natural numbers* k *such that* $1 \leq k < |\underline{c}_1|$, *it is true that*

$$c_{2,k} < c_{2,k+1}$$

Definition 661 *If*

$$\underline{c}_1 = \left\{c_{1,1}, c_{1,2}, \ldots, c_{1,k}, \ldots, c_{1,|\underline{c}_1|}\right\}$$

is a chroma set in a well-formed pitch system then the function $\underline{\mathrm{c}} \downarrow (\underline{c}_1)$ *returns the unique ordered chroma set*

$$\underline{\mathrm{c}} \downarrow (\underline{c}_1) = \left[c_{2,1}, c_{2,2}, \ldots, c_{2,k}, \ldots, c_{2,|\underline{c}_1|}\right]$$

that satisfies the following conditions:

1. $(c \in \underline{\mathrm{c}} \downarrow (\underline{c}_1)) \iff (c \in \underline{c}_1)$;
2. $|\underline{\mathrm{c}} \downarrow (\underline{c}_1)| = |\underline{c}_1|$;
3. *For all natural numbers* k *such that* $1 \leq k < |\underline{c}_1|$, *it is true that*

$$c_{2,k} > c_{2,k+1}$$

Sorting morph sets

Definition 662 *If*

$$\underline{m}_1 = \left\{m_{1,1}, m_{1,2}, \ldots, m_{1,k}, \ldots, m_{1,|\underline{m}_1|}\right\}$$

is a morph set in a well-formed pitch system then the function $\underline{\mathrm{m}} \uparrow (\underline{m}_1)$ *returns the unique ordered morph set*

$$\underline{\mathrm{m}} \uparrow (\underline{m}_1) = \left[m_{2,1}, m_{2,2}, \ldots, m_{2,k}, \ldots, m_{2,|\underline{m}_1|}\right]$$

that satisfies the following conditions:

1. $(m \in \underline{\mathrm{m}} \uparrow (\underline{m}_1)) \iff (m \in \underline{m}_1)$;
2. $|\underline{\mathrm{m}} \uparrow (\underline{m}_1)| = |\underline{m}_1|$;
3. *For all natural numbers* k *such that* $1 \leq k < |\underline{m}_1|$, *it is true that*

$$m_{2,k} < m_{2,k+1}$$

Definition 663 *If*

$$\underline{m}_1 = \left\{m_{1,1}, m_{1,2}, \ldots, m_{1,k}, \ldots, m_{1,|\underline{m}_1|}\right\}$$

is a morph set in a well-formed pitch system then the function $\underline{\mathrm{m}} \downarrow (\underline{m}_1)$ *returns the unique ordered morph set*

$$\underline{\mathrm{m}} \downarrow (\underline{m}_1) = \left[m_{2,1}, m_{2,2}, \ldots, m_{2,k}, \ldots, m_{2,|\underline{m}_1|}\right]$$

that satisfies the following conditions:

1. $(m \in \underline{\mathrm{m}} \downarrow (\underline{m}_1)) \iff (m \in \underline{m}_1)$;
2. $|\underline{\mathrm{m}} \downarrow (\underline{m}_1)| = |\underline{m}_1|$;
3. *For all natural numbers k such that $1 \leq k < |\underline{m}_1|$, it is true that*

$$m_{2,k} > m_{2,k+1}$$

Sorting chromamorph sets

Definition 664 *If*

$$\underline{q}_1 = \left\{q_{1,1}, q_{1,2}, \ldots, q_{1,k}, \ldots, q_{1,|\underline{q}_1|}\right\}$$

is a chromamorph set in a well-formed pitch system then the function $\underline{\mathrm{q}} \uparrow_{\mathrm{c}} \left(\underline{q}_1\right)$ *returns the unique ordered chromamorph set*

$$\underline{\mathrm{q}} \uparrow_{\mathrm{c}} \left(\underline{q}_1\right) = \left[q_{2,1}, q_{2,2}, \ldots, q_{2,k}, \ldots, q_{2,|\underline{q}_1|}\right]$$

that satisfies the following conditions:

1. $\left(q \in \underline{\mathrm{q}} \uparrow_{\mathrm{c}} \left(\underline{q}_1\right)\right) \iff \left(q \in \underline{q}_1\right)$;
2. $\left|\underline{\mathrm{q}} \uparrow_{\mathrm{c}} \left(\underline{q}_1\right)\right| = \left|\underline{q}_1\right|$;
3. *For all natural numbers k such that $1 \leq k < |\underline{q}_1|$, it is true that*

$$q_{2,k} \leq_{\mathrm{c}} q_{2,k+1}$$

4. *For all natural numbers k such that $1 \leq k < |\underline{q}_1|$, it is true that*

$$(q_{2,k} \equiv_{\mathrm{c}} q_{2,k+1}) \Rightarrow (q_{2,k} <_{\mathrm{m}} q_{2,k+1})$$

Definition 665 *If*

$$\underline{q}_1 = \left\{q_{1,1}, q_{1,2}, \ldots, q_{1,k}, \ldots, q_{1,|\underline{q}_1|}\right\}$$

is a chromamorph set in a well-formed pitch system then the function $\underline{\mathrm{q}} \downarrow_{\mathrm{c}} \left(\underline{q}_1\right)$ *returns the unique ordered chromamorph set*

$$\underline{\mathrm{q}} \downarrow_{\mathrm{c}} \left(\underline{q}_1\right) = \left[q_{2,1}, q_{2,2}, \ldots, q_{2,k}, \ldots, q_{2,|\underline{q}_1|}\right]$$

that satisfies the following conditions:

1. $\left(q \in \underline{\mathrm{q}} \downarrow_{\mathrm{c}} \left(\underline{q}_1\right)\right) \iff \left(q \in \underline{q}_1\right)$;
2. $\left|\underline{\mathrm{q}} \downarrow_{\mathrm{c}} \left(\underline{q}_1\right)\right| = \left|\underline{q}_1\right|$;

3. *For all natural numbers k such that $1 \leq k < |\underline{q}_1|$, it is true that*

$$q_{2,k} \geq_{\mathrm{c}} q_{2,k+1}$$

4. *For all natural numbers k such that $1 \leq k < |\underline{q}_1|$, it is true that*

$$(q_{2,k} \equiv_{\mathrm{c}} q_{2,k+1}) \Rightarrow (q_{2,k} >_{\mathrm{m}} q_{2,k+1})$$

Definition 666 *If*

$$\underline{q}_1 = \left\{q_{1,1}, q_{1,2}, \ldots, q_{1,k}, \ldots, q_{1,|\underline{q}_1|}\right\}$$

is a chromamorph set in a well-formed pitch system then the function $\underline{\mathrm{q}} \uparrow_{\mathrm{m}} \left(\underline{q}_1\right)$ returns the unique ordered chromamorph set

$$\underline{\mathrm{q}} \uparrow_{\mathrm{m}} \left(\underline{q}_1\right) = \left[q_{2,1}, q_{2,2}, \ldots, q_{2,k}, \ldots, q_{2,|\underline{q}_1|}\right]$$

that satisfies the following conditions:

1. $\left(q \in \underline{\mathrm{q}} \uparrow_{\mathrm{m}} \left(\underline{q}_1\right)\right) \iff \left(q \in \underline{q}_1\right)$;
2. $\left|\underline{\mathrm{q}} \uparrow_{\mathrm{m}} \left(\underline{q}_1\right)\right| = \left|\underline{q}_1\right|$;
3. *For all natural numbers k such that $1 \leq k < |\underline{q}_1|$, it is true that*

$$q_{2,k} \leq_{\mathrm{m}} q_{2,k+1}$$

4. *For all natural numbers k such that $1 \leq k < |\underline{q}_1|$, it is true that*

$$(q_{2,k} \equiv_{\mathrm{m}} q_{2,k+1}) \Rightarrow (q_{2,k} <_{\mathrm{c}} q_{2,k+1})$$

Definition 667 *If*

$$\underline{q}_1 = \left\{q_{1,1}, q_{1,2}, \ldots, q_{1,k}, \ldots, q_{1,|\underline{q}_1|}\right\}$$

is a chromamorph set in a well-formed pitch system then the function $\underline{\mathrm{q}} \downarrow_{\mathrm{m}} \left(\underline{q}_1\right)$ returns the unique ordered chromamorph set

$$\underline{\mathrm{q}} \downarrow_{\mathrm{m}} \left(\underline{q}_1\right) = \left[q_{2,1}, q_{2,2}, \ldots, q_{2,k}, \ldots, q_{2,|\underline{q}_1|}\right]$$

that satisfies the following conditions:

1. $\left(q \in \underline{\mathrm{q}} \downarrow_{\mathrm{m}} \left(\underline{q}_1\right)\right) \iff \left(q \in \underline{q}_1\right)$;
2. $\left|\underline{\mathrm{q}} \downarrow_{\mathrm{m}} \left(\underline{q}_1\right)\right| = \left|\underline{q}_1\right|$;
3. *For all natural numbers k such that $1 \leq k < |\underline{q}_1|$, it is true that*

$$q_{2,k} \geq_{\mathrm{m}} q_{2,k+1}$$

4. *For all natural numbers k such that $1 \leq k < |\underline{q}_1|$, it is true that*

$$(q_{2,k} \equiv_{\mathrm{m}} q_{2,k+1}) \Rightarrow (q_{2,k} >_{\mathrm{c}} q_{2,k+1})$$

Sorting chromatic genus sets

Definition 668 *If*

$$\underline{g}_{\mathrm{c},1} = \left\{ g_{\mathrm{c},1,1}, g_{\mathrm{c},1,2}, \ldots, g_{\mathrm{c},1,k}, \ldots, g_{\mathrm{c},1,|\underline{g}_{\mathrm{c},1}|} \right\}$$

is a chromatic genus set in a well-formed pitch system then the function $\underline{\mathrm{g}}_{\mathrm{c}} \uparrow \left(\underline{g}_{\mathrm{c},1}\right)$ *returns the unique ordered chromatic genus set*

$$\underline{\mathrm{g}}_{\mathrm{c}} \uparrow \left(\underline{g}_{\mathrm{c},1}\right) = \left[g_{\mathrm{c},2,1}, g_{\mathrm{c},2,2}, \ldots, g_{\mathrm{c},2,k}, \ldots, g_{\mathrm{c},2,|\underline{g}_{\mathrm{c},1}|} \right]$$

that satisfies the following conditions:

1. $\left(g_{\mathrm{c}} \in \underline{\mathrm{g}}_{\mathrm{c}} \uparrow \left(\underline{g}_{\mathrm{c},1}\right)\right) \iff \left(g_{\mathrm{c}} \in \underline{g}_{\mathrm{c},1}\right)$;
2. $\left|\underline{\mathrm{g}}_{\mathrm{c}} \uparrow \left(\underline{g}_{\mathrm{c},1}\right)\right| = \left|\underline{g}_{\mathrm{c},1}\right|$;
3. *For all natural numbers k such that $1 \leq k < |\underline{g}_{\mathrm{c},1}|$, it is true that*

$$g_{\mathrm{c},2,k} < g_{\mathrm{c},2,k+1}$$

Definition 669 *If*

$$\underline{g}_{\mathrm{c},1} = \left\{ g_{\mathrm{c},1,1}, g_{\mathrm{c},1,2}, \ldots, g_{\mathrm{c},1,k}, \ldots, g_{\mathrm{c},1,|\underline{g}_{\mathrm{c},1}|} \right\}$$

is a chromatic genus set in a well-formed pitch system then the function $\underline{\mathrm{g}}_{\mathrm{c}} \downarrow \left(\underline{g}_{\mathrm{c},1}\right)$ *returns the unique ordered chromatic genus set*

$$\underline{\mathrm{g}}_{\mathrm{c}} \downarrow \left(\underline{g}_{\mathrm{c},1}\right) = \left[g_{\mathrm{c},2,1}, g_{\mathrm{c},2,2}, \ldots, g_{\mathrm{c},2,k}, \ldots, g_{\mathrm{c},2,|\underline{g}_{\mathrm{c},1}|} \right]$$

that satisfies the following conditions:

1. $\left(g_{\mathrm{c}} \in \underline{\mathrm{g}}_{\mathrm{c}} \downarrow \left(\underline{g}_{\mathrm{c},1}\right)\right) \iff \left(g_{\mathrm{c}} \in \underline{g}_{\mathrm{c},1}\right)$;
2. $\left|\underline{\mathrm{g}}_{\mathrm{c}} \downarrow \left(\underline{g}_{\mathrm{c},1}\right)\right| = \left|\underline{g}_{\mathrm{c},1}\right|$;
3. *For all natural numbers k such that $1 \leq k < |\underline{g}_{\mathrm{c},1}|$, it is true that*

$$g_{\mathrm{c},2,k} > g_{\mathrm{c},2,k+1}$$

Sorting genus sets

Definition 670 *If*

$$\underline{g}_{1} = \left\{ g_{1,1}, g_{1,2}, \ldots, g_{1,k}, \ldots, g_{1,|\underline{g}_{1}|} \right\}$$

is a genus set in a well-formed pitch system then the function $\underline{\mathrm{g}} \uparrow_{\mathrm{g_c}} \left(\underline{g}_{1}\right)$ *returns the unique ordered genus set*

$$\underline{\mathrm{g}} \uparrow_{\mathrm{g_c}} \left(\underline{g}_{1}\right) = \left[g_{2,1}, g_{2,2}, \ldots, g_{2,k}, \ldots, g_{2,|\underline{g}_{1}|} \right]$$

that satisfies the following conditions:

1. $\left(g \in \underline{\mathrm{g}} \uparrow_{\mathrm{g_c}} \left(\underline{g}_{1}\right)\right) \iff \left(g \in \underline{g}_{1}\right)$;
2. $\left|\underline{\mathrm{g}} \uparrow_{\mathrm{g_c}} \left(\underline{g}_{1}\right)\right| = \left|\underline{g}_{1}\right|$;
3. *For all natural numbers k such that $1 \leq k < |\underline{g}_{1}|$, it is true that*

$$g_{2,k} \leq_{\mathrm{g_c}} g_{2,k+1}$$

4. *For all natural numbers k such that $1 \leq k < |\underline{g}_1|$, it is true that*

$$(g_{2,k} \equiv_{\mathrm{gc}} g_{2,k+1}) \Rightarrow (g_{2,k} <_{\mathrm{m}} g_{2,k+1})$$

Definition 671 *If*

$$\underline{g}_1 = \left\{g_{1,1}, g_{1,2}, \ldots, g_{1,k}, \ldots, g_{1,|\underline{g}_1|}\right\}$$

is a genus set in a well-formed pitch system then the function $\underline{\mathrm{g}} \downarrow_{\mathrm{gc}} \left(\underline{g}_1\right)$ returns the unique ordered genus set

$$\underline{\mathrm{g}} \downarrow_{\mathrm{gc}} \left(\underline{g}_1\right) = \left[g_{2,1}, g_{2,2}, \ldots, g_{2,k}, \ldots, g_{2,|\underline{g}_1|}\right]$$

that satisfies the following conditions:

1. $\left(g \in \underline{\mathrm{g}} \downarrow_{\mathrm{gc}} \left(\underline{g}_1\right)\right) \iff \left(g \in \underline{g}_1\right)$*;*
2. $\left|\underline{\mathrm{g}} \downarrow_{\mathrm{gc}} \left(\underline{g}_1\right)\right| = \left|\underline{g}_1\right|$*;*
3. *For all natural numbers k such that $1 \leq k < |\underline{g}_1|$, it is true that*

$$g_{2,k} \geq_{\mathrm{gc}} g_{2,k+1}$$

4. *For all natural numbers k such that $1 \leq k < |\underline{g}_1|$, it is true that*

$$(g_{2,k} \equiv_{\mathrm{gc}} g_{2,k+1}) \Rightarrow (g_{2,k} >_{\mathrm{m}} g_{2,k+1})$$

Definition 672 *If*

$$\underline{g}_1 = \left\{g_{1,1}, g_{1,2}, \ldots, g_{1,k}, \ldots, g_{1,|\underline{g}_1|}\right\}$$

is a genus set in a well-formed pitch system then the function $\underline{\mathrm{g}} \uparrow_{\mathrm{m}} \left(\underline{g}_1\right)$ returns the unique ordered genus set

$$\underline{\mathrm{g}} \uparrow_{\mathrm{m}} \left(\underline{g}_1\right) = \left[g_{2,1}, g_{2,2}, \ldots, g_{2,k}, \ldots, g_{2,|\underline{g}_1|}\right]$$

that satisfies the following conditions:

1. $\left(g \in \underline{\mathrm{g}} \uparrow_{\mathrm{m}} \left(\underline{g}_1\right)\right) \iff \left(g \in \underline{g}_1\right)$*;*
2. $\left|\underline{\mathrm{g}} \uparrow_{\mathrm{m}} \left(\underline{g}_1\right)\right| = \left|\underline{g}_1\right|$*;*
3. *For all natural numbers k such that $1 \leq k < |\underline{g}_1|$, it is true that*

$$g_{2,k} \leq_{\mathrm{m}} g_{2,k+1}$$

4. *For all natural numbers k such that $1 \leq k < |\underline{g}_1|$, it is true that*

$$(g_{2,k} \equiv_{\mathrm{m}} g_{2,k+1}) \Rightarrow (g_{2,k} <_{\mathrm{gc}} g_{2,k+1})$$

Definition 673 *If*

$$\underline{g}_1 = \left\{g_{1,1}, g_{1,2}, \ldots, g_{1,k}, \ldots, g_{1,|\underline{g}_1|}\right\}$$

is a genus set in a well-formed pitch system then the function $\underline{\mathrm{g}} \downarrow_{\mathrm{m}} \left(\underline{g}_1\right)$ returns the unique ordered genus set

$$\underline{\mathrm{g}} \downarrow_{\mathrm{m}} \left(\underline{g}_1\right) = \left[g_{2,1}, g_{2,2}, \ldots, g_{2,k}, \ldots, g_{2,|\underline{g}_1|}\right]$$

that satisfies the following conditions:

1. $\left(g \in \underline{\mathrm{g}} \downarrow_{\mathrm{m}} \left(\underline{g}_1\right)\right) \iff \left(g \in \underline{g}_1\right)$;
2. $\left|\underline{\mathrm{g}} \downarrow_{\mathrm{m}} \left(\underline{g}_1\right)\right| = \left|\underline{g}_1\right|$;
3. *For all natural numbers k such that $1 \leq k < |\underline{g}_1|$, it is true that*

$$g_{2,k} \geq_{\mathrm{m}} g_{2,k+1}$$

4. *For all natural numbers k such that $1 \leq k < |\underline{g}_1|$, it is true that*

$$(g_{2,k} \equiv_{\mathrm{m}} g_{2,k+1}) \Rightarrow (g_{2,k} >_{\mathrm{g_c}} g_{2,k+1})$$

4.7.7 Inequalities between *MIPS* object sets

Inequalities between pitch sets

Definition 674 *If $\underline{p}_1$ and $\underline{p}_2$ are any two pitch sets in a pitch system ψ then $\underline{p}_1$ is* chromatic pitch less than *$\underline{p}_2$, denoted*

$$\underline{p}_1 <_{\mathrm{p_c}} \underline{p}_2$$

if and only if one of the following conditions is satisfied:

1. $\mathrm{e}\left(\underline{\mathrm{p}} \uparrow_{\mathrm{p_c}} \left(\underline{p}_1\right), 1\right) <_{\mathrm{p_c}} \mathrm{e}\left(\underline{\mathrm{p}} \uparrow_{\mathrm{p_c}} \left(\underline{p}_2\right), 1\right)$
2. *There exists a value n such that*

$$\left(\mathrm{e}\left(\underline{\mathrm{p}} \uparrow_{\mathrm{p_c}} \left(\underline{p}_1\right), k\right) = \mathrm{e}\left(\underline{\mathrm{p}} \uparrow_{\mathrm{p_c}} \left(\underline{p}_2\right), k\right) \forall k : 1 \leq k \leq n\right)$$
$$\wedge$$
$$\left(\mathrm{e}\left(\underline{\mathrm{p}} \uparrow_{\mathrm{p_c}} \left(\underline{p}_1\right), n+1\right) <_{\mathrm{p_c}} \mathrm{e}\left(\underline{\mathrm{p}} \uparrow_{\mathrm{p_c}} \left(\underline{p}_2\right), n+1\right)\right)$$

Definition 675 *If $\underline{p}_1$ and $\underline{p}_2$ are any two pitch sets in a pitch system ψ then $\underline{p}_1$ is* chromatic pitch greater than *$\underline{p}_2$, denoted*

$$\underline{p}_1 >_{\mathrm{p_c}} \underline{p}_2$$

if and only if one of the following conditions is satisfied:

1. $\mathrm{e}\left(\underline{\mathrm{p}} \uparrow_{\mathrm{p_c}} \left(\underline{p}_1\right), 1\right) >_{\mathrm{p_c}} \mathrm{e}\left(\underline{\mathrm{p}} \uparrow_{\mathrm{p_c}} \left(\underline{p}_2\right), 1\right)$
2. *There exists a value n such that*

$$\left(\mathrm{e}\left(\underline{\mathrm{p}} \uparrow_{\mathrm{p_c}} \left(\underline{p}_1\right), k\right) = \mathrm{e}\left(\underline{\mathrm{p}} \uparrow_{\mathrm{p_c}} \left(\underline{p}_2\right), k\right) \forall k : 1 \leq k \leq n\right)$$
$$\wedge$$
$$\left(\mathrm{e}\left(\underline{\mathrm{p}} \uparrow_{\mathrm{p_c}} \left(\underline{p}_1\right), n+1\right) >_{\mathrm{p_c}} \mathrm{e}\left(\underline{\mathrm{p}} \uparrow_{\mathrm{p_c}} \left(\underline{p}_2\right), n+1\right)\right)$$

Definition 676 *If $\underline{p}_1$ and $\underline{p}_2$ are any two pitch sets in a pitch system ψ then $\underline{p}_1$ is* morphetic pitch less than *$\underline{p}_2$, denoted*

$$\underline{p}_1 <_{\mathrm{p_m}} \underline{p}_2$$

if and only if one of the following conditions is satisfied:

1. $\mathrm{e}\left(\underline{\mathrm{p}} \uparrow_{\mathrm{p_m}} \left(\underline{p}_1\right), 1\right) <_{\mathrm{p_m}} \mathrm{e}\left(\underline{\mathrm{p}} \uparrow_{\mathrm{p_m}} \left(\underline{p}_2\right), 1\right)$

2. *There exists a value n such that*

$$\left(\mathrm{e}\left(\underline{\mathrm{p}}\uparrow_{\mathrm{pm}}\left(\underline{p}_1\right),k\right)=\mathrm{e}\left(\underline{\mathrm{p}}\uparrow_{\mathrm{pm}}\left(\underline{p}_2\right),k\right)\forall k:1\le k\le n\right)$$
$$\wedge$$
$$\left(\mathrm{e}\left(\underline{\mathrm{p}}\uparrow_{\mathrm{pm}}\left(\underline{p}_1\right),n+1\right)<_{\mathrm{pm}}\mathrm{e}\left(\underline{\mathrm{p}}\uparrow_{\mathrm{pm}}\left(\underline{p}_2\right),n+1\right)\right)$$

Definition 677 *If $\underline{p}_1$ and $\underline{p}_2$ are any two pitch sets in a pitch system ψ then $\underline{p}_1$ is* morphetic pitch greater than $\underline{p}_2$*, denoted*

$$\underline{p}_1>_{\mathrm{pm}}\underline{p}_2$$

if and only if one of the following conditions is satisfied:

1. $\mathrm{e}\left(\underline{\mathrm{p}}\uparrow_{\mathrm{pm}}\left(\underline{p}_1\right),1\right)>_{\mathrm{pm}}\mathrm{e}\left(\underline{\mathrm{p}}\uparrow_{\mathrm{pm}}\left(\underline{p}_2\right),1\right)$
2. *There exists a value n such that*

$$\left(\mathrm{e}\left(\underline{\mathrm{p}}\uparrow_{\mathrm{pm}}\left(\underline{p}_1\right),k\right)=\mathrm{e}\left(\underline{\mathrm{p}}\uparrow_{\mathrm{pm}}\left(\underline{p}_2\right),k\right)\forall k:1\le k\le n\right)$$
$$\wedge$$
$$\left(\mathrm{e}\left(\underline{\mathrm{p}}\uparrow_{\mathrm{pm}}\left(\underline{p}_1\right),n+1\right)>_{\mathrm{pm}}\mathrm{e}\left(\underline{\mathrm{p}}\uparrow_{\mathrm{pm}}\left(\underline{p}_2\right),n+1\right)\right)$$

Definition 678 *If $\underline{p}_1$ and $\underline{p}_2$ are any two pitch sets in a pitch system ψ then $\underline{p}_1$ is* chromatic pitch less than or equal to $\underline{p}_2$*, denoted*

$$\underline{p}_1\le_{\mathrm{pc}}\underline{p}_2$$

if and only if

$$\left(\underline{p}_1=\underline{p}_2\right)\vee\left(\underline{p}_1<_{\mathrm{pc}}\underline{p}_2\right)$$

Definition 679 *If $\underline{p}_1$ and $\underline{p}_2$ are any two pitch sets in a pitch system ψ then $\underline{p}_1$ is* chromatic pitch greater than or equal to $\underline{p}_2$*, denoted*

$$\underline{p}_1\ge_{\mathrm{pc}}\underline{p}_2$$

if and only if

$$\left(\underline{p}_1=\underline{p}_2\right)\vee\left(\underline{p}_1>_{\mathrm{pc}}\underline{p}_2\right)$$

Definition 680 *If $\underline{p}_1$ and $\underline{p}_2$ are any two pitch sets in a pitch system ψ then $\underline{p}_1$ is* morphetic pitch less than or equal to $\underline{p}_2$*, denoted*

$$\underline{p}_1\le_{\mathrm{pm}}\underline{p}_2$$

if and only if

$$\left(\underline{p}_1=\underline{p}_2\right)\vee\left(\underline{p}_1<_{\mathrm{pm}}\underline{p}_2\right)$$

Definition 681 *If $\underline{p}_1$ and $\underline{p}_2$ are any two pitch sets in a pitch system ψ then $\underline{p}_1$ is* morphetic pitch greater than or equal to $\underline{p}_2$*, denoted*

$$\underline{p}_1\ge_{\mathrm{pm}}\underline{p}_2$$

if and only if

$$\left(\underline{p}_1=\underline{p}_2\right)\vee\left(\underline{p}_1>_{\mathrm{pm}}\underline{p}_2\right)$$

Inequalities between chromatic pitch sets

Definition 682 *If* $\underline{p}_{\mathrm{c},1}$ *and* $\underline{p}_{\mathrm{c},2}$ *are any two chromatic pitch sets in a pitch system* ψ *then* $\underline{p}_{\mathrm{c},1}$ *is* less than $\underline{p}_{\mathrm{c},2}$*, denoted*

$$\underline{p}_{\mathrm{c},1} < \underline{p}_{\mathrm{c},2}$$

if and only if one of the following conditions is satisfied:

1. $\mathrm{e}\left(\underline{\mathrm{p}}_{\mathrm{c}} \uparrow \left(\underline{p}_{\mathrm{c},1}\right), 1\right) < \mathrm{e}\left(\underline{\mathrm{p}}_{\mathrm{c}} \uparrow \left(\underline{p}_{\mathrm{c},2}\right), 1\right)$
2. *There exists a value* n *such that*

$$\left(\mathrm{e}\left(\underline{\mathrm{p}}_{\mathrm{c}} \uparrow \left(\underline{p}_{\mathrm{c},1}\right), k\right) = \mathrm{e}\left(\underline{\mathrm{p}}_{\mathrm{c}} \uparrow \left(\underline{p}_{\mathrm{c},2}\right), k\right) \forall k : 1 \leq k \leq n\right)$$
$$\wedge$$
$$\left(\mathrm{e}\left(\underline{\mathrm{p}}_{\mathrm{c}} \uparrow \left(\underline{p}_{\mathrm{c},1}\right), n+1\right) < \mathrm{e}\left(\underline{\mathrm{p}}_{\mathrm{c}} \uparrow \left(\underline{p}_{\mathrm{c},2}\right), n+1\right)\right)$$

Definition 683 *If* $\underline{p}_{\mathrm{c},1}$ *and* $\underline{p}_{\mathrm{c},2}$ *are any two chromatic pitch sets in a pitch system* ψ *then* $\underline{p}_{\mathrm{c},1}$ *is* greater than $\underline{p}_{\mathrm{c},2}$*, denoted*

$$\underline{p}_{\mathrm{c},1} > \underline{p}_{\mathrm{c},2}$$

if and only if one of the following conditions is satisfied:

1. $\mathrm{e}\left(\underline{\mathrm{p}}_{\mathrm{c}} \uparrow \left(\underline{p}_{\mathrm{c},1}\right), 1\right) > \mathrm{e}\left(\underline{\mathrm{p}}_{\mathrm{c}} \uparrow \left(\underline{p}_{\mathrm{c},2}\right), 1\right)$
2. *There exists a value* n *such that*

$$\left(\mathrm{e}\left(\underline{\mathrm{p}}_{\mathrm{c}} \uparrow \left(\underline{p}_{\mathrm{c},1}\right), k\right) = \mathrm{e}\left(\underline{\mathrm{p}}_{\mathrm{c}} \uparrow \left(\underline{p}_{\mathrm{c},2}\right), k\right) \forall k : 1 \leq k \leq n\right)$$
$$\wedge$$
$$\left(\mathrm{e}\left(\underline{\mathrm{p}}_{\mathrm{c}} \uparrow \left(\underline{p}_{\mathrm{c},1}\right), n+1\right) > \mathrm{e}\left(\underline{\mathrm{p}}_{\mathrm{c}} \uparrow \left(\underline{p}_{\mathrm{c},2}\right), n+1\right)\right)$$

Definition 684 *If* $\underline{p}_{\mathrm{c},1}$ *and* $\underline{p}_{\mathrm{c},2}$ *are any two chromatic pitch sets in a pitch system* ψ *then* $\underline{p}_{\mathrm{c},1}$ *is* less than or equal to $\underline{p}_{\mathrm{c},2}$*, denoted*

$$\underline{p}_{\mathrm{c},1} \leq \underline{p}_{\mathrm{c},2}$$

if and only if

$$\left(\underline{p}_{\mathrm{c},1} = \underline{p}_{\mathrm{c},2}\right) \vee \left(\underline{p}_{\mathrm{c},1} < \underline{p}_{\mathrm{c},2}\right)$$

Definition 685 *If* $\underline{p}_{\mathrm{c},1}$ *and* $\underline{p}_{\mathrm{c},2}$ *are any two chromatic pitch sets in a pitch system* ψ *then* $\underline{p}_{\mathrm{c},1}$ *is* greater than or equal to $\underline{p}_{\mathrm{c},2}$*, denoted*

$$\underline{p}_{\mathrm{c},1} \geq \underline{p}_{\mathrm{c},2}$$

if and only if

$$\left(\underline{p}_{\mathrm{c},1} = \underline{p}_{\mathrm{c},2}\right) \vee \left(\underline{p}_{\mathrm{c},1} > \underline{p}_{\mathrm{c},2}\right)$$

Inequalities between morphetic pitch sets

Definition 686 *If* $\underline{p}_{\mathrm{m},1}$ *and* $\underline{p}_{\mathrm{m},2}$ *are any two morphetic pitch sets in a pitch system* ψ *then* $\underline{p}_{\mathrm{m},1}$ *is* less than $\underline{p}_{\mathrm{m},2}$*, denoted*

$$\underline{p}_{\mathrm{m},1} < \underline{p}_{\mathrm{m},2}$$

if and only if one of the following conditions is satisfied:

1. $\mathrm{e}\left(\underline{\mathrm{p}}_{\mathrm{m}} \uparrow \left(\underline{p}_{\mathrm{m},1}\right), 1\right) < \mathrm{e}\left(\underline{\mathrm{p}}_{\mathrm{m}} \uparrow \left(\underline{p}_{\mathrm{m},2}\right), 1\right)$
2. *There exists a value n such that*

$$\left(\mathrm{e}\left(\underline{\mathrm{p}}_{\mathrm{m}} \uparrow \left(\underline{p}_{\mathrm{m},1}\right), k\right) = \mathrm{e}\left(\underline{\mathrm{p}}_{\mathrm{m}} \uparrow \left(\underline{p}_{\mathrm{m},2}\right), k\right) \forall k : 1 \leq k \leq n\right)$$
$$\wedge$$
$$\left(\mathrm{e}\left(\underline{\mathrm{p}}_{\mathrm{m}} \uparrow \left(\underline{p}_{\mathrm{m},1}\right), n+1\right) < \mathrm{e}\left(\underline{\mathrm{p}}_{\mathrm{m}} \uparrow \left(\underline{p}_{\mathrm{m},2}\right), n+1\right)\right)$$

Definition 687 *If $\underline{p}_{\mathrm{m},1}$ and $\underline{p}_{\mathrm{m},2}$ are any two morphetic pitch sets in a pitch system ψ then $\underline{p}_{\mathrm{m},1}$ is* greater than $\underline{p}_{\mathrm{m},2}$, *denoted*

$$\underline{p}_{\mathrm{m},1} > \underline{p}_{\mathrm{m},2}$$

if and only if one of the following conditions is satisfied:

1. $\mathrm{e}\left(\underline{\mathrm{p}}_{\mathrm{m}} \uparrow \left(\underline{p}_{\mathrm{m},1}\right), 1\right) > \mathrm{e}\left(\underline{\mathrm{p}}_{\mathrm{m}} \uparrow \left(\underline{p}_{\mathrm{m},2}\right), 1\right)$
2. *There exists a value n such that*

$$\left(\mathrm{e}\left(\underline{\mathrm{p}}_{\mathrm{m}} \uparrow \left(\underline{p}_{\mathrm{m},1}\right), k\right) = \mathrm{e}\left(\underline{\mathrm{p}}_{\mathrm{m}} \uparrow \left(\underline{p}_{\mathrm{m},2}\right), k\right) \forall k : 1 \leq k \leq n\right)$$
$$\wedge$$
$$\left(\mathrm{e}\left(\underline{\mathrm{p}}_{\mathrm{m}} \uparrow \left(\underline{p}_{\mathrm{m},1}\right), n+1\right) > \mathrm{e}\left(\underline{\mathrm{p}}_{\mathrm{m}} \uparrow \left(\underline{p}_{\mathrm{m},2}\right), n+1\right)\right)$$

Definition 688 *If $\underline{p}_{\mathrm{m},1}$ and $\underline{p}_{\mathrm{m},2}$ are any two morphetic pitch sets in a pitch system ψ then $\underline{p}_{\mathrm{m},1}$ is* less than or equal to $\underline{p}_{\mathrm{m},2}$, *denoted*

$$\underline{p}_{\mathrm{m},1} \leq \underline{p}_{\mathrm{m},2}$$

if and only if

$$\left(\underline{p}_{\mathrm{m},1} = \underline{p}_{\mathrm{m},2}\right) \vee \left(\underline{p}_{\mathrm{m},1} < \underline{p}_{\mathrm{m},2}\right)$$

Definition 689 *If $\underline{p}_{\mathrm{m},1}$ and $\underline{p}_{\mathrm{m},2}$ are any two morphetic pitch sets in a pitch system ψ then $\underline{p}_{\mathrm{m},1}$ is* greater than or equal to $\underline{p}_{\mathrm{m},2}$, *denoted*

$$\underline{p}_{\mathrm{m},1} \geq \underline{p}_{\mathrm{m},2}$$

if and only if

$$\left(\underline{p}_{\mathrm{m},1} = \underline{p}_{\mathrm{m},2}\right) \vee \left(\underline{p}_{\mathrm{m},1} > \underline{p}_{\mathrm{m},2}\right)$$

Inequalities between frequency sets

Definition 690 *If $\underline{f}_1$ and $\underline{f}_2$ are any two frequency sets in a pitch system ψ then $\underline{f}_1$ is* less than $\underline{f}_2$, *denoted*

$$\underline{f}_1 < \underline{f}_2$$

if and only if one of the following conditions is satisfied:

1. $\mathrm{e}\left(\underline{\mathrm{f}} \uparrow \left(\underline{f}_1\right), 1\right) < \mathrm{e}\left(\underline{\mathrm{f}} \uparrow \left(\underline{f}_2\right), 1\right)$
2. *There exists a value n such that*

$$\left(\mathrm{e}\left(\underline{\mathrm{f}} \uparrow \left(\underline{f}_1\right), k\right) = \mathrm{e}\left(\underline{\mathrm{f}} \uparrow \left(\underline{f}_2\right), k\right) \forall k : 1 \leq k \leq n\right)$$
$$\wedge$$
$$\left(\mathrm{e}\left(\underline{\mathrm{f}} \uparrow \left(\underline{f}_1\right), n+1\right) < \mathrm{e}\left(\underline{\mathrm{f}} \uparrow \left(\underline{f}_2\right), n+1\right)\right)$$

Definition 691 *If* $\underline{f}_1$ *and* $\underline{f}_2$ *are any two frequency sets in a pitch system* ψ *then* $\underline{f}_1$ *is* greater than $\underline{f}_2$, *denoted*

$$\underline{f}_1 > \underline{f}_2$$

if and only if one of the following conditions is satisfied:

1. $\mathrm{e}\left(\underline{\mathrm{f}} \uparrow \left(\underline{f}_1\right), 1\right) > \mathrm{e}\left(\underline{\mathrm{f}} \uparrow \left(\underline{f}_2\right), 1\right)$
2. *There exists a value* n *such that*

$$\left(\mathrm{e}\left(\underline{\mathrm{f}} \uparrow \left(\underline{f}_1\right), k\right) = \mathrm{e}\left(\underline{\mathrm{f}} \uparrow \left(\underline{f}_2\right), k\right) \forall k : 1 \leq k \leq n\right)$$
$$\wedge$$
$$\left(\mathrm{e}\left(\underline{\mathrm{f}} \uparrow \left(\underline{f}_1\right), n+1\right) > \mathrm{e}\left(\underline{\mathrm{f}} \uparrow \left(\underline{f}_2\right), n+1\right)\right)$$

Definition 692 *If* $\underline{f}_1$ *and* $\underline{f}_2$ *are any two frequency sets in a pitch system* ψ *then* $\underline{f}_1$ *is* less than or equal to $\underline{f}_2$, *denoted*

$$\underline{f}_1 \leq \underline{f}_2$$

if and only if

$$\left(\underline{f}_1 = \underline{f}_2\right) \vee \left(\underline{f}_1 < \underline{f}_2\right)$$

Definition 693 *If* $\underline{f}_1$ *and* $\underline{f}_2$ *are any two frequency sets in a pitch system* ψ *then* $\underline{f}_1$ *is* greater than or equal to $\underline{f}_2$, *denoted*

$$\underline{f}_1 \geq \underline{f}_2$$

if and only if

$$\left(\underline{f}_1 = \underline{f}_2\right) \vee \left(\underline{f}_1 > \underline{f}_2\right)$$

Inequalities between chroma sets

Definition 694 *If* $\underline{c}_1$ *and* $\underline{c}_2$ *are any two chroma sets in a pitch system* ψ *then* $\underline{c}_1$ *is* less than $\underline{c}_2$, *denoted*

$$\underline{c}_1 < \underline{c}_2$$

if and only if one of the following conditions is satisfied:

1. $\mathrm{e}\left(\underline{\mathrm{c}} \uparrow (\underline{c}_1), 1\right) < \mathrm{e}\left(\underline{\mathrm{c}} \uparrow (\underline{c}_2), 1\right)$
2. *There exists a value* n *such that*

$$\left(\mathrm{e}\left(\underline{\mathrm{c}} \uparrow (\underline{c}_1), k\right) = \mathrm{e}\left(\underline{\mathrm{c}} \uparrow (\underline{c}_2), k\right) \forall k : 1 \leq k \leq n\right)$$
$$\wedge$$
$$\left(\mathrm{e}\left(\underline{\mathrm{c}} \uparrow (\underline{c}_1), n+1\right) < \mathrm{e}\left(\underline{\mathrm{c}} \uparrow (\underline{c}_2), n+1\right)\right)$$

Definition 695 *If* $\underline{c}_1$ *and* $\underline{c}_2$ *are any two chroma sets in a pitch system* ψ *then* $\underline{c}_1$ *is* greater than $\underline{c}_2$, *denoted*

$$\underline{c}_1 > \underline{c}_2$$

if and only if one of the following conditions is satisfied:

1. $\mathrm{e}\left(\underline{\mathrm{c}} \uparrow (\underline{c}_1), 1\right) > \mathrm{e}\left(\underline{\mathrm{c}} \uparrow (\underline{c}_2), 1\right)$

2. *There exists a value n such that*

$$\begin{gathered}(\mathrm{e}\,(\underline{\mathrm{c}} \uparrow (\underline{c}_1), k) = \mathrm{e}\,(\underline{\mathrm{c}} \uparrow (\underline{c}_2), k)\, \forall k : 1 \leq k \leq n) \\ \wedge \\ (\mathrm{e}\,(\underline{\mathrm{c}} \uparrow (\underline{c}_1), n+1) > \mathrm{e}\,(\underline{\mathrm{c}} \uparrow (\underline{c}_2), n+1))\end{gathered}$$

Definition 696 *If $\underline{c}_1$ and $\underline{c}_2$ are any two chroma sets in a pitch system ψ then $\underline{c}_1$ is* less than or equal to $\underline{c}_2$, *denoted*

$$\underline{c}_1 \leq \underline{c}_2$$

if and only if

$$(\underline{c}_1 = \underline{c}_2) \vee (\underline{c}_1 < \underline{c}_2)$$

Definition 697 *If $\underline{c}_1$ and $\underline{c}_2$ are any two chroma sets in a pitch system ψ then $\underline{c}_1$ is* greater than or equal to $\underline{c}_2$, *denoted*

$$\underline{c}_1 \geq \underline{c}_2$$

if and only if

$$(\underline{c}_1 = \underline{c}_2) \vee (\underline{c}_1 > \underline{c}_2)$$

Inequalities between morph sets

Definition 698 *If $\underline{m}_1$ and $\underline{m}_2$ are any two morph sets in a pitch system ψ then $\underline{m}_1$ is* less than $\underline{m}_2$, *denoted*

$$\underline{m}_1 < \underline{m}_2$$

if and only if one of the following conditions is satisfied:

1. $\mathrm{e}\,(\underline{\mathrm{m}} \uparrow (\underline{m}_1), 1) < \mathrm{e}\,(\underline{\mathrm{m}} \uparrow (\underline{m}_2), 1)$

2. *There exists a value n such that*

$$\begin{gathered}(\mathrm{e}\,(\underline{\mathrm{m}} \uparrow (\underline{m}_1), k) = \mathrm{e}\,(\underline{\mathrm{m}} \uparrow (\underline{m}_2), k)\, \forall k : 1 \leq k \leq n) \\ \wedge \\ (\mathrm{e}\,(\underline{\mathrm{m}} \uparrow (\underline{m}_1), n+1) < \mathrm{e}\,(\underline{\mathrm{m}} \uparrow (\underline{m}_2), n+1))\end{gathered}$$

Definition 699 *If $\underline{m}_1$ and $\underline{m}_2$ are any two morph sets in a pitch system ψ then $\underline{m}_1$ is* greater than $\underline{m}_2$, *denoted*

$$\underline{m}_1 > \underline{m}_2$$

if and only if one of the following conditions is satisfied:

1. $\mathrm{e}\,(\underline{\mathrm{m}} \uparrow (\underline{m}_1), 1) > \mathrm{e}\,(\underline{\mathrm{m}} \uparrow (\underline{m}_2), 1)$

2. *There exists a value n such that*

$$\begin{gathered}(\mathrm{e}\,(\underline{\mathrm{m}} \uparrow (\underline{m}_1), k) = \mathrm{e}\,(\underline{\mathrm{m}} \uparrow (\underline{m}_2), k)\, \forall k : 1 \leq k \leq n) \\ \wedge \\ (\mathrm{e}\,(\underline{\mathrm{m}} \uparrow (\underline{m}_1), n+1) > \mathrm{e}\,(\underline{\mathrm{m}} \uparrow (\underline{m}_2), n+1))\end{gathered}$$

Definition 700 *If* $\underline{m}_1$ *and* $\underline{m}_2$ *are any two morph sets in a pitch system* ψ *then* $\underline{m}_1$ *is* less than or equal to $\underline{m}_2$*, denoted*

$$\underline{m}_1 \leq \underline{m}_2$$

if and only if

$$(\underline{m}_1 = \underline{m}_2) \vee (\underline{m}_1 < \underline{m}_2)$$

Definition 701 *If* $\underline{m}_1$ *and* $\underline{m}_2$ *are any two morph sets in a pitch system* ψ *then* $\underline{m}_1$ *is* greater than or equal to $\underline{m}_2$*, denoted*

$$\underline{m}_1 \geq \underline{m}_2$$

if and only if

$$(\underline{m}_1 = \underline{m}_2) \vee (\underline{m}_1 > \underline{m}_2)$$

Inequalities between chromamorph sets

Definition 702 *If* $\underline{q}_1$ *and* $\underline{q}_2$ *are any two chromamorph sets in a pitch system* ψ *then* $\underline{q}_1$ *is* chroma less than $\underline{q}_2$*, denoted*

$$\underline{q}_1 <_{\mathrm{c}} \underline{q}_2$$

if and only if one of the following conditions is satisfied:

1. $\mathrm{e}\left(\underline{\mathrm{q}} \uparrow_{\mathrm{c}} \left(\underline{q}_1\right), 1\right) <_{\mathrm{c}} \mathrm{e}\left(\underline{\mathrm{q}} \uparrow_{\mathrm{c}} \left(\underline{q}_2\right), 1\right)$
2. *There exists a value* n *such that*

$$\left(\mathrm{e}\left(\underline{\mathrm{q}} \uparrow_{\mathrm{c}} \left(\underline{q}_1\right), k\right) = \mathrm{e}\left(\underline{\mathrm{q}} \uparrow_{\mathrm{c}} \left(\underline{q}_2\right), k\right) \forall k : 1 \leq k \leq n\right)$$
$$\wedge$$
$$\left(\mathrm{e}\left(\underline{\mathrm{q}} \uparrow_{\mathrm{c}} \left(\underline{q}_1\right), n+1\right) <_{\mathrm{c}} \mathrm{e}\left(\underline{\mathrm{q}} \uparrow_{\mathrm{c}} \left(\underline{q}_2\right), n+1\right)\right)$$

Definition 703 *If* $\underline{q}_1$ *and* $\underline{q}_2$ *are any two chromamorph sets in a pitch system* ψ *then* $\underline{q}_1$ *is* chroma greater than $\underline{q}_2$*, denoted*

$$\underline{q}_1 >_{\mathrm{c}} \underline{q}_2$$

if and only if one of the following conditions is satisfied:

1. $\mathrm{e}\left(\underline{\mathrm{q}} \uparrow_{\mathrm{c}} \left(\underline{q}_1\right), 1\right) >_{\mathrm{c}} \mathrm{e}\left(\underline{\mathrm{q}} \uparrow_{\mathrm{c}} \left(\underline{q}_2\right), 1\right)$
2. *There exists a value* n *such that*

$$\left(\mathrm{e}\left(\underline{\mathrm{q}} \uparrow_{\mathrm{c}} \left(\underline{q}_1\right), k\right) = \mathrm{e}\left(\underline{\mathrm{q}} \uparrow_{\mathrm{c}} \left(\underline{q}_2\right), k\right) \forall k : 1 \leq k \leq n\right)$$
$$\wedge$$
$$\left(\mathrm{e}\left(\underline{\mathrm{q}} \uparrow_{\mathrm{c}} \left(\underline{q}_1\right), n+1\right) >_{\mathrm{c}} \mathrm{e}\left(\underline{\mathrm{q}} \uparrow_{\mathrm{c}} \left(\underline{q}_2\right), n+1\right)\right)$$

Definition 704 *If* $\underline{q}_1$ *and* $\underline{q}_2$ *are any two chromamorph sets in a pitch system* ψ *then* $\underline{q}_1$ *is* morph less than $\underline{q}_2$*, denoted*

$$\underline{q}_1 <_{\mathrm{m}} \underline{q}_2$$

if and only if one of the following conditions is satisfied:

1. $\mathrm{e}\left(\underline{\mathrm{q}} \uparrow_{\mathrm{m}} \left(\underline{q}_1\right), 1\right) <_{\mathrm{m}} \mathrm{e}\left(\underline{\mathrm{q}} \uparrow_{\mathrm{m}} \left(\underline{q}_2\right), 1\right)$

2. *There exists a value n such that*

$$\left(\mathrm{e}\left(\underline{\mathrm{q}}\uparrow_{\mathrm{m}}\left(\underline{q}_1\right),k\right)=\mathrm{e}\left(\underline{\mathrm{q}}\uparrow_{\mathrm{m}}\left(\underline{q}_2\right),k\right)\forall k:1\leq k\leq n\right)$$
$$\wedge$$
$$\left(\mathrm{e}\left(\underline{\mathrm{q}}\uparrow_{\mathrm{m}}\left(\underline{q}_1\right),n+1\right)<_{\mathrm{m}}\mathrm{e}\left(\underline{\mathrm{q}}\uparrow_{\mathrm{m}}\left(\underline{q}_2\right),n+1\right)\right)$$

Definition 705 *If $\underline{q}_1$ and $\underline{q}_2$ are any two chromamorph sets in a pitch system ψ then $\underline{q}_1$ is* morph greater than $\underline{q}_2$*, denoted*

$$\underline{q}_1 >_{\mathrm{m}} \underline{q}_2$$

if and only if one of the following conditions is satisfied:

1. $\mathrm{e}\left(\underline{\mathrm{q}}\uparrow_{\mathrm{m}}\left(\underline{q}_1\right),1\right)>_{\mathrm{m}}\mathrm{e}\left(\underline{\mathrm{q}}\uparrow_{\mathrm{m}}\left(\underline{q}_2\right),1\right)$

2. *There exists a value n such that*

$$\left(\mathrm{e}\left(\underline{\mathrm{q}}\uparrow_{\mathrm{m}}\left(\underline{q}_1\right),k\right)=\mathrm{e}\left(\underline{\mathrm{q}}\uparrow_{\mathrm{m}}\left(\underline{q}_2\right),k\right)\forall k:1\leq k\leq n\right)$$
$$\wedge$$
$$\left(\mathrm{e}\left(\underline{\mathrm{q}}\uparrow_{\mathrm{m}}\left(\underline{q}_1\right),n+1\right)>_{\mathrm{m}}\mathrm{e}\left(\underline{\mathrm{q}}\uparrow_{\mathrm{m}}\left(\underline{q}_2\right),n+1\right)\right)$$

Definition 706 *If $\underline{q}_1$ and $\underline{q}_2$ are any two chromamorph sets in a pitch system ψ then $\underline{q}_1$ is* chroma less than or equal to $\underline{q}_2$*, denoted*

$$\underline{q}_1 \leq_{\mathrm{c}} \underline{q}_2$$

if and only if

$$\left(\underline{q}_1=\underline{q}_2\right)\vee\left(\underline{q}_1<_{\mathrm{c}}\underline{q}_2\right)$$

Definition 707 *If $\underline{q}_1$ and $\underline{q}_2$ are any two chromamorph sets in a pitch system ψ then $\underline{q}_1$ is* chroma greater than or equal to $\underline{q}_2$*, denoted*

$$\underline{q}_1 \geq_{\mathrm{c}} \underline{q}_2$$

if and only if

$$\left(\underline{q}_1=\underline{q}_2\right)\vee\left(\underline{q}_1>_{\mathrm{c}}\underline{q}_2\right)$$

Definition 708 *If $\underline{q}_1$ and $\underline{q}_2$ are any two chromamorph sets in a pitch system ψ then $\underline{q}_1$ is* morph less than or equal to $\underline{q}_2$*, denoted*

$$\underline{q}_1 \leq_{\mathrm{m}} \underline{q}_2$$

if and only if

$$\left(\underline{q}_1=\underline{q}_2\right)\vee\left(\underline{q}_1<_{\mathrm{m}}\underline{q}_2\right)$$

Definition 709 *If $\underline{q}_1$ and $\underline{q}_2$ are any two chromamorph sets in a pitch system ψ then $\underline{q}_1$ is* morph greater than or equal to $\underline{q}_2$*, denoted*

$$\underline{q}_1 \geq_{\mathrm{m}} \underline{q}_2$$

if and only if

$$\left(\underline{q}_1=\underline{q}_2\right)\vee\left(\underline{q}_1>_{\mathrm{m}}\underline{q}_2\right)$$

Inequalities between chromatic genus sets

Definition 710 *If* $\underline{g}_{c,1}$ *and* $\underline{g}_{c,2}$ *are any two chromatic genus sets in a pitch system* ψ *then* $\underline{g}_{c,1}$ *is* less than $\underline{g}_{c,2}$*, denoted*

$$\underline{g}_{c,1} < \underline{g}_{c,2}$$

if and only if one of the following conditions is satisfied:

1. $\mathrm{e}\left(\underline{\mathrm{g}}_{\mathrm{c}} \uparrow \left(\underline{g}_{c,1}\right), 1\right) < \mathrm{e}\left(\underline{\mathrm{g}}_{\mathrm{c}} \uparrow \left(\underline{g}_{c,2}\right), 1\right)$
2. *There exists a value* n *such that*

$$\left(\mathrm{e}\left(\underline{\mathrm{g}}_{\mathrm{c}} \uparrow \left(\underline{g}_{c,1}\right), k\right) = \mathrm{e}\left(\underline{\mathrm{g}}_{\mathrm{c}} \uparrow \left(\underline{g}_{c,2}\right), k\right) \forall k : 1 \leq k \leq n\right)$$
$$\wedge$$
$$\left(\mathrm{e}\left(\underline{\mathrm{g}}_{\mathrm{c}} \uparrow \left(\underline{g}_{c,1}\right), n+1\right) < \mathrm{e}\left(\underline{\mathrm{g}}_{\mathrm{c}} \uparrow \left(\underline{g}_{c,2}\right), n+1\right)\right)$$

Definition 711 *If* $\underline{g}_{c,1}$ *and* $\underline{g}_{c,2}$ *are any two chromatic genus sets in a pitch system* ψ *then* $\underline{g}_{c,1}$ *is* greater than $\underline{g}_{c,2}$*, denoted*

$$\underline{g}_{c,1} > \underline{g}_{c,2}$$

if and only if one of the following conditions is satisfied:

1. $\mathrm{e}\left(\underline{\mathrm{g}}_{\mathrm{c}} \uparrow \left(\underline{g}_{c,1}\right), 1\right) > \mathrm{e}\left(\underline{\mathrm{g}}_{\mathrm{c}} \uparrow \left(\underline{g}_{c,2}\right), 1\right)$
2. *There exists a value* n *such that*

$$\left(\mathrm{e}\left(\underline{\mathrm{g}}_{\mathrm{c}} \uparrow \left(\underline{g}_{c,1}\right), k\right) = \mathrm{e}\left(\underline{\mathrm{g}}_{\mathrm{c}} \uparrow \left(\underline{g}_{c,2}\right), k\right) \forall k : 1 \leq k \leq n\right)$$
$$\wedge$$
$$\left(\mathrm{e}\left(\underline{\mathrm{g}}_{\mathrm{c}} \uparrow \left(\underline{g}_{c,1}\right), n+1\right) > \mathrm{e}\left(\underline{\mathrm{g}}_{\mathrm{c}} \uparrow \left(\underline{g}_{c,2}\right), n+1\right)\right)$$

Definition 712 *If* $\underline{g}_{c,1}$ *and* $\underline{g}_{c,2}$ *are any two chromatic genus sets in a pitch system* ψ *then* $\underline{g}_{c,1}$ *is* less than or equal to $\underline{g}_{c,2}$*, denoted*

$$\underline{g}_{c,1} \leq \underline{g}_{c,2}$$

if and only if

$$\left(\underline{g}_{c,1} = \underline{g}_{c,2}\right) \vee \left(\underline{g}_{c,1} < \underline{g}_{c,2}\right)$$

Definition 713 *If* $\underline{g}_{c,1}$ *and* $\underline{g}_{c,2}$ *are any two chromatic genus sets in a pitch system* ψ *then* $\underline{g}_{c,1}$ *is* greater than or equal to $\underline{g}_{c,2}$*, denoted*

$$\underline{g}_{c,1} \geq \underline{g}_{c,2}$$

if and only if

$$\left(\underline{g}_{c,1} = \underline{g}_{c,2}\right) \vee \left(\underline{g}_{c,1} > \underline{g}_{c,2}\right)$$

Inequalities between genus sets

Definition 714 *If* $\underline{g}_1$ *and* $\underline{g}_2$ *are any two genus sets in a pitch system* ψ *then* $\underline{g}_1$ *is* chromatic genus less than $\underline{g}_2$*, denoted*

$$\underline{g}_1 <_{\mathrm{g_c}} \underline{g}_2$$

if and only if one of the following conditions is satisfied:

1. $\mathrm{e}\left(\underline{\mathrm{g}}\uparrow_{\mathrm{gc}}\left(\underline{g}_1\right),1\right) <_{\mathrm{gc}} \mathrm{e}\left(\underline{\mathrm{g}}\uparrow_{\mathrm{gc}}\left(\underline{g}_2\right),1\right)$
2. *There exists a value n such that*

$$\left(\mathrm{e}\left(\underline{\mathrm{g}}\uparrow_{\mathrm{gc}}\left(\underline{g}_1\right),k\right) = \mathrm{e}\left(\underline{\mathrm{g}}\uparrow_{\mathrm{gc}}\left(\underline{g}_2\right),k\right) \forall k : 1 \leq k \leq n\right)$$
$$\wedge$$
$$\left(\mathrm{e}\left(\underline{\mathrm{g}}\uparrow_{\mathrm{gc}}\left(\underline{g}_1\right),n+1\right) <_{\mathrm{gc}} \mathrm{e}\left(\underline{\mathrm{g}}\uparrow_{\mathrm{gc}}\left(\underline{g}_2\right),n+1\right)\right)$$

Definition 715 *If $\underline{g}_1$ and $\underline{g}_2$ are any two genus sets in a pitch system ψ then $\underline{g}_1$ is* chromatic genus greater than $\underline{g}_2$*, denoted*

$$\underline{g}_1 >_{\mathrm{gc}} \underline{g}_2$$

if and only if one of the following conditions is satisfied:

1. $\mathrm{e}\left(\underline{\mathrm{g}}\uparrow_{\mathrm{gc}}\left(\underline{g}_1\right),1\right) >_{\mathrm{gc}} \mathrm{e}\left(\underline{\mathrm{g}}\uparrow_{\mathrm{gc}}\left(\underline{g}_2\right),1\right)$
2. *There exists a value n such that*

$$\left(\mathrm{e}\left(\underline{\mathrm{g}}\uparrow_{\mathrm{gc}}\left(\underline{g}_1\right),k\right) = \mathrm{e}\left(\underline{\mathrm{g}}\uparrow_{\mathrm{gc}}\left(\underline{g}_2\right),k\right) \forall k : 1 \leq k \leq n\right)$$
$$\wedge$$
$$\left(\mathrm{e}\left(\underline{\mathrm{g}}\uparrow_{\mathrm{gc}}\left(\underline{g}_1\right),n+1\right) >_{\mathrm{gc}} \mathrm{e}\left(\underline{\mathrm{g}}\uparrow_{\mathrm{gc}}\left(\underline{g}_2\right),n+1\right)\right)$$

Definition 716 *If $\underline{g}_1$ and $\underline{g}_2$ are any two genus sets in a pitch system ψ then $\underline{g}_1$ is* morph less than $\underline{g}_2$*, denoted*

$$\underline{g}_1 <_{\mathrm{m}} \underline{g}_2$$

if and only if one of the following conditions is satisfied:

1. $\mathrm{e}\left(\underline{\mathrm{g}}\uparrow_{\mathrm{m}}\left(\underline{g}_1\right),1\right) <_{\mathrm{m}} \mathrm{e}\left(\underline{\mathrm{g}}\uparrow_{\mathrm{m}}\left(\underline{g}_2\right),1\right)$
2. *There exists a value n such that*

$$\left(\mathrm{e}\left(\underline{\mathrm{g}}\uparrow_{\mathrm{m}}\left(\underline{g}_1\right),k\right) = \mathrm{e}\left(\underline{\mathrm{g}}\uparrow_{\mathrm{m}}\left(\underline{g}_2\right),k\right) \forall k : 1 \leq k \leq n\right)$$
$$\wedge$$
$$\left(\mathrm{e}\left(\underline{\mathrm{g}}\uparrow_{\mathrm{m}}\left(\underline{g}_1\right),n+1\right) <_{\mathrm{m}} \mathrm{e}\left(\underline{\mathrm{g}}\uparrow_{\mathrm{m}}\left(\underline{g}_2\right),n+1\right)\right)$$

Definition 717 *If $\underline{g}_1$ and $\underline{g}_2$ are any two genus sets in a pitch system ψ then $\underline{g}_1$ is* morph greater than $\underline{g}_2$*, denoted*

$$\underline{g}_1 >_{\mathrm{m}} \underline{g}_2$$

if and only if one of the following conditions is satisfied:

1. $\mathrm{e}\left(\underline{\mathrm{g}}\uparrow_{\mathrm{m}}\left(\underline{g}_1\right),1\right) >_{\mathrm{m}} \mathrm{e}\left(\underline{\mathrm{g}}\uparrow_{\mathrm{m}}\left(\underline{g}_2\right),1\right)$
2. *There exists a value n such that*

$$\left(\mathrm{e}\left(\underline{\mathrm{g}}\uparrow_{\mathrm{m}}\left(\underline{g}_1\right),k\right) = \mathrm{e}\left(\underline{\mathrm{g}}\uparrow_{\mathrm{m}}\left(\underline{g}_2\right),k\right) \forall k : 1 \leq k \leq n\right)$$
$$\wedge$$
$$\left(\mathrm{e}\left(\underline{\mathrm{g}}\uparrow_{\mathrm{m}}\left(\underline{g}_1\right),n+1\right) >_{\mathrm{m}} \mathrm{e}\left(\underline{\mathrm{g}}\uparrow_{\mathrm{m}}\left(\underline{g}_2\right),n+1\right)\right)$$

Definition 718 *If* $\underline{g}_1$ *and* $\underline{g}_2$ *are any two genus sets in a pitch system* ψ *then* $\underline{g}_1$ *is* chromatic genus less than or equal to $\underline{g}_2$*, denoted*

$$\underline{g}_1 \leq_{\mathrm{gc}} \underline{g}_2$$

if and only if

$$\left(\underline{g}_1 = \underline{g}_2\right) \vee \left(\underline{g}_1 <_{\mathrm{gc}} \underline{g}_2\right)$$

Definition 719 *If* $\underline{g}_1$ *and* $\underline{g}_2$ *are any two genus sets in a pitch system* ψ *then* $\underline{g}_1$ *is* chromatic genus greater than or equal to $\underline{g}_2$*, denoted*

$$\underline{g}_1 \geq_{\mathrm{gc}} \underline{g}_2$$

if and only if

$$\left(\underline{g}_1 = \underline{g}_2\right) \vee \left(\underline{g}_1 >_{\mathrm{gc}} \underline{g}_2\right)$$

Definition 720 *If* $\underline{g}_1$ *and* $\underline{g}_2$ *are any two genus sets in a pitch system* ψ *then* $\underline{g}_1$ *is* morph less than or equal to $\underline{g}_2$*, denoted*

$$\underline{g}_1 \leq_{\mathrm{m}} \underline{g}_2$$

if and only if

$$\left(\underline{g}_1 = \underline{g}_2\right) \vee \left(\underline{g}_1 <_{\mathrm{m}} \underline{g}_2\right)$$

Definition 721 *If* $\underline{g}_1$ *and* $\underline{g}_2$ *are any two genus sets in a pitch system* ψ *then* $\underline{g}_1$ *is* morph greater than or equal to $\underline{g}_2$*, denoted*

$$\underline{g}_1 \geq_{\mathrm{m}} \underline{g}_2$$

if and only if

$$\left(\underline{g}_1 = \underline{g}_2\right) \vee \left(\underline{g}_1 >_{\mathrm{m}} \underline{g}_2\right)$$

4.8 Sets of *MIPS* intervals

4.8.1 Universal sets of *MIPS* intervals

Definition 722 *The* universal set of chromatic pitch intervals $\underline{\Delta p}_{\mathrm{c,u}}$ *for a specified pitch system* ψ *is the set that contains all and only chromatic pitch intervals within* ψ*.*

Theorem 723 *For a specified pitch system* ψ*,*

$$\underline{\Delta p}_{\mathrm{c,u}} = \mathbb{Z}$$

where $\mathbb{Z}$ *is the universal set of integers.*

Proof

R1	Let		$\Delta p = [\Delta p_{\mathrm{c}}, \Delta p_{\mathrm{m}}]$ be any pitch interval whatsoever in a pitch system ψ.
R2	R1 & 237	$\Rightarrow$	Δp_{c} can only take any integer value.
R3	R2 & 722	$\Rightarrow$	$\underline{\Delta p}_{\mathrm{c,u}} = \mathbb{Z}$ where $\mathbb{Z}$ is the universal set of integers.

Definition 724 *The* universal set of morphetic pitch intervals $\underline{\Delta p}_{\mathrm{m,u}}$ *for a specified pitch system* ψ *is the set that contains all and only morphetic pitch intervals within* ψ*.*

Theorem 725 *For a specified pitch system ψ,*

$$\underline{\Delta p}_{\mathrm{m,u}} = \mathbb{Z}$$

where $\mathbb{Z}$ is the universal set of integers.
Proof

R1	Let		$\Delta p = [\Delta p_{\mathrm{c}}, \Delta p_{\mathrm{m}}]$ be any pitch interval whatsoever in a pitch system ψ.
R2	R1 & 241	$\Rightarrow$	Δp_{m} can only take any integer value.
R3	R2 & 724	$\Rightarrow$	$\underline{\Delta p}_{\mathrm{m,u}} = \mathbb{Z}$ where $\mathbb{Z}$ is the universal set of integers.

Definition 726 *The* universal set of pitch intervals $\underline{\Delta p}_{\mathrm{u}}$ *for a specified pitch system ψ is the set that contains all and only pitch intervals within ψ.*

Theorem 727 *For a specified pitch system ψ, $\underline{\Delta p}_{\mathrm{u}}$ contains all and only those values*

$$\Delta p = [\Delta p_{\mathrm{c}}, \Delta p_{\mathrm{m}}]$$

such that

$$\left(\Delta p_{\mathrm{c}} \in \underline{\Delta p}_{\mathrm{c,u}}\right) \wedge \left(\Delta p_{\mathrm{m}} \in \underline{\Delta p}_{\mathrm{m,u}}\right)$$

Proof

R1	Let		$\Delta p = [\Delta p_{\mathrm{c}}, \Delta p_{\mathrm{m}}]$ be any pitch interval whatsoever in a pitch system ψ.
R2	R1 & 722	$\Rightarrow$	Δp_{c} can only take any value such that $\Delta p_{\mathrm{c}} \in \underline{\Delta p}_{\mathrm{c,u}}$.
R3	R1 & 724	$\Rightarrow$	Δp_{m} can only take any value such that $\Delta p_{\mathrm{m}} \in \underline{\Delta p}_{\mathrm{m,u}}$.
R4	R1, R2, R3 & 726	$\Rightarrow$	$\underline{\Delta p}_{\mathrm{u}}$ contains all and only those values $\Delta p = [\Delta p_{\mathrm{c}}, \Delta p_{\mathrm{m}}]$ such that $\left(\Delta p_{\mathrm{c}} \in \underline{\Delta p}_{\mathrm{c,u}}\right) \wedge \left(\Delta p_{\mathrm{m}} \in \underline{\Delta p}_{\mathrm{m,u}}\right)$.

Definition 728 *The* universal set of frequency intervals $\underline{\Delta f}_{\mathrm{u}}$ *for a specified pitch system ψ is the set that contains all and only those values that can be taken by a frequency interval in ψ.*

Theorem 729 *For a specified pitch system ψ,*

$$\underline{\Delta f}_{\mathrm{u}} = \mathbb{R}^{+}$$

where $\mathbb{R}^{+}$ is the universal set of real numbers greater than zero.
Proof

R1	Let		$\Delta f = \Delta\,\mathrm{f}\,(f_1, f_2)$ where f_1 and f_2 are any two frequencies in a pitch system ψ.
R2	R1 & 243	$\Rightarrow$	Δf can only take any positive real value.
R3	R2 & 728	$\Rightarrow$	$\underline{\Delta f}_{\mathrm{u}} = \mathbb{R}^{+}$

Definition 730 *The* universal set of chroma intervals $\underline{\Delta c}_{\mathrm{u}}$ *for a specified pitch system* ψ *is the set that contains all and only those values that can be taken by a chroma interval in* ψ*.*

Theorem 731 *For a specified pitch system*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

$\underline{\Delta c}_{\mathrm{u}}$ *contains all and only those values* Δc *such that*

$$(\Delta c \in \mathbb{Z}) \wedge (0 \leq \Delta c < \mu_{\mathrm{c}})$$

where $\mathbb{Z}$ *is the universal set of integers.*

Proof

R1	Let		$\Delta c = \Delta\,\mathrm{c}\,(c_1, c_2)$ where c_1 and c_2 are any two chromae in ψ.
R2	R1 & 214	$\Rightarrow$	Δc can only take any value such that $(\Delta c \in \mathbb{Z}) \wedge (0 \leq \Delta c < \mu_{\mathrm{c}})$.
R3	R1, R2 & 730	$\Rightarrow$	$\underline{\Delta c}_{\mathrm{u}}$ contains all and only those values Δc such that $(\Delta c \in \mathbb{Z}) \wedge (0 \leq \Delta c < \mu_{\mathrm{c}})$ where $\mathbb{Z}$ is the universal set of integers.

Definition 732 *The* universal set of morph intervals $\underline{\Delta m}_{\mathrm{u}}$ *for a specified pitch system* ψ *is the set that contains all and only those values that can be taken by a morph interval in* ψ*.*

Theorem 733 *For a specified pitch system*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

$\underline{\Delta m}_{\mathrm{u}}$ *contains all and only those values* Δm *such that*

$$(\Delta m \in \mathbb{Z}) \wedge (0 \leq \Delta m < \mu_{\mathrm{m}})$$

where $\mathbb{Z}$ *is the universal set of integers.*

Proof

R1	Let		$\Delta m = \Delta\,\mathrm{m}\,(m_1, m_2)$ where m_1 and m_2 are any two morphs in ψ.
R2	R1 & 218	$\Rightarrow$	Δm can only take any value such that $(\Delta m \in \mathbb{Z}) \wedge (0 \leq \Delta m < \mu_{\mathrm{m}})$.
R3	R1, R2 & 732	$\Rightarrow$	$\underline{\Delta m}_{\mathrm{u}}$ contains all and only those values Δm such that $(\Delta m \in \mathbb{Z}) \wedge (0 \leq \Delta m < \mu_{\mathrm{m}})$ where $\mathbb{Z}$ is the universal set of integers.

Definition 734 *The* universal set of chromamorph intervals $\underline{\Delta q}_{\mathrm{u}}$ *for a specified pitch system* ψ *is the set that contains all and only those values that can be taken by a chromamorph interval in* ψ.

Theorem 735 *For a specified pitch system*

$$\psi = [\mu_{\mathrm{c}}, \mu_{\mathrm{m}}, f_0, p_{\mathrm{c},0}]$$

$\underline{\Delta q}_{\mathrm{u}}$ *contains all and only those values*

$$\Delta q = [\Delta c, \Delta m]$$

such that

$$(\Delta m \in \underline{\Delta m}_{\mathrm{u}}) \wedge (\Delta c \in \underline{\Delta c}_{\mathrm{u}})$$

Proof

R1	Let		$\Delta q = [\Delta c, \Delta m]$ be any chromamorph interval whatsoever in a pitch system ψ.
R2	R1 & 730	$\Rightarrow$	Δc can only take any value such that $\Delta c \in \underline{\Delta c}_{\mathrm{u}}$.
R3	R1 & 732	$\Rightarrow$	Δm can only take any value such that $\Delta m \in \underline{\Delta m}_{\mathrm{u}}$.
R4	R1, R2, R3 & 734	$\Rightarrow$	$\underline{\Delta q}_{\mathrm{u}}$ contains all and only those values $\Delta q = [\Delta c, \Delta m]$ such that $(\Delta c \in \underline{\Delta c}_{\mathrm{u}}) \wedge (\Delta m \in \underline{\Delta m}_{\mathrm{u}})$.

Definition 736 *The* universal set of chromatic genus intervals $\underline{\Delta g}_{\mathrm{c,u}}$ *for a specified pitch system* ψ *is the set that contains all and only those values that can be taken by a chromatic genus interval in* ψ.

Theorem 737 *For a specified pitch system* ψ, $\underline{\Delta g}_{\mathrm{c,u}} = \mathbb{Z}$ *where* $\mathbb{Z}$ *is the universal set of integers.*

Proof

R1	Let		$\Delta g_{\mathrm{c}} = \Delta\,\mathrm{g_c}\,(p_1, p_2)$ where p_1 and p_2 are any two pitches in ψ.
R2	R1 & 256	$\Rightarrow$	Δg_{c} can only take any integer value.
R3	736 & R2	$\Rightarrow$	$\underline{\Delta g}_{\mathrm{c,u}} = \mathbb{Z}$ where $\mathbb{Z}$ is the universal set of integers.

Definition 738 *The* universal set of genus intervals $\underline{\Delta g}_{\mathrm{u}}$ *for a specified pitch system* ψ *is the set that contains all and only those values that can be taken by a genus interval in* ψ.

Theorem 739 *For a specified pitch system* ψ, $\underline{\Delta g}_{\mathrm{u}}$ *contains all and only those values*

$$\Delta g = [\Delta g_{\mathrm{c}}, \Delta m]$$

such that

$$(\Delta m \in \underline{\Delta m}_{\mathrm{u}}) \wedge \left(\Delta g_{\mathrm{c}} \in \underline{\Delta g}_{\mathrm{c,u}}\right)$$

Proof

R1	Let		$\Delta g = [\Delta g_{\mathrm{c}}, \Delta m]$ be any genus interval whatsoever in a pitch system ψ.
R2	R1 & 736	$\Rightarrow$	Δg_{c} can only take any value such that $\Delta g_{\mathrm{c}} \in \underline{\Delta g}_{\mathrm{c,u}}$.
R3	R1 & 732	$\Rightarrow$	Δm can only take any value such that $\Delta m \in \underline{\Delta m}_{\mathrm{u}}$.
R4	R1, R2, R3 & 738	$\Rightarrow$	$\underline{\Delta g}_{\mathrm{u}}$ contains all and only those values $\Delta g = [\Delta g_{\mathrm{c}}, \Delta m]$ such that $\left(\Delta g_{\mathrm{c}} \in \underline{\Delta g}_{\mathrm{c,u}}\right) \wedge (\Delta m \in \underline{\Delta m}_{\mathrm{u}})$.

4.8.2 Definitions for sets of *MIPS* intervals

Definition 740 *If* $\underline{\Delta p}_{\mathrm{u}}$ *is the universal set of pitch intervals for the pitch system* ψ*, then* $\underline{\Delta p}$ *is a well-formed* pitch interval set *in* ψ *if and only if*

$$\underline{\Delta p} \subseteq \underline{\Delta p}_{\mathrm{u}}$$

Definition 741 *If* $\underline{\Delta p}_{\mathrm{c,u}}$ *is the universal set of chromatic pitch intervals for the pitch system* ψ*, then* $\underline{\Delta p}_{\mathrm{c}}$ *is a well-formed* chromatic pitch interval set *in* ψ *if and only if*

$$\underline{\Delta p}_{\mathrm{c}} \subseteq \underline{\Delta p}_{\mathrm{c,u}}$$

Definition 742 *If* $\underline{\Delta p}_{\mathrm{m,u}}$ *is the universal set of morphetic pitch intervals for the pitch system* ψ*, then* $\underline{\Delta p}_{\mathrm{m}}$ *is a well-formed* morphetic pitch interval set *in* ψ *if and only if*

$$\underline{\Delta p}_{\mathrm{m}} \subseteq \underline{\Delta p}_{\mathrm{m,u}}$$

Definition 743 *If* $\underline{\Delta f}_{\mathrm{u}}$ *is the universal set of frequency intervals for the pitch system* ψ*, then* $\underline{\Delta f}$ *is a well-formed* frequency interval set *in* ψ *if and only if*

$$\underline{\Delta f} \subseteq \underline{\Delta f}_{\mathrm{u}}$$

Definition 744 *If* $\underline{\Delta c}_{\mathrm{u}}$ *is the universal set of chroma intervals for the pitch system* ψ*, then* $\underline{\Delta c}$ *is a well-formed* chroma interval set *in* ψ *if and only if*

$$\underline{\Delta c} \subseteq \underline{\Delta c}_{\mathrm{u}}$$

Definition 745 *If* $\underline{\Delta m}_{\mathrm{u}}$ *is the universal set of morph intervals for the pitch system* ψ*, then* $\underline{\Delta m}$ *is a well-formed* morph interval set *in* ψ *if and only if*

$$\underline{\Delta m} \subseteq \underline{\Delta m}_{\mathrm{u}}$$

Definition 746 *If* $\underline{\Delta q}_{\mathrm{u}}$ *is the universal set of chromamorph intervals for the pitch system* ψ*, then* $\underline{\Delta q}$ *is a well-formed* chromamorph interval set *in* ψ *if and only if*

$$\underline{\Delta q} \subseteq \underline{\Delta q}_{\mathrm{u}}$$

Definition 747 *If* $\underline{\Delta g}_{\mathrm{c,u}}$ *is the universal set of chromatic genus intervals for the pitch system* ψ*, then* $\underline{\Delta g}_{\mathrm{c}}$ *is a well-formed* chromatic genus interval set *in* ψ *if and only if*

$$\underline{\Delta g}_{\mathrm{c}} \subseteq \underline{\Delta g}_{\mathrm{c,u}}$$

Definition 748 *If* $\underline{\Delta g}_{\mathrm{u}}$ *is the universal set of genus intervals for the pitch system* ψ*, then* $\underline{\Delta g}$ *is a well-formed* genus interval set *in* ψ *if and only if*

$$\underline{\Delta g} \subseteq \underline{\Delta g}_{\mathrm{u}}$$

4.8.3 Derived *MIPS* interval sets

Deriving *MIPS* interval sets from a pitch interval set

Definition 749 *If*

$$\underline{\Delta p} = \{\Delta p_1, \Delta p_2, \ldots \Delta p_k, \ldots\}$$

is a pitch interval set in a pitch system ψ*, then the following function returns the chromatic pitch interval set of* $\underline{\Delta p}$*:*

$$\underline{\Delta\,\mathrm{p}}_{\mathrm{c}}\left(\underline{\Delta p}\right) = \bigcup_{k=1}^{|\underline{\Delta p}|} \{\Delta\,\mathrm{p_c}\left(\Delta p_k\right)\}$$

Definition 750 *If*

$$\underline{\Delta p} = \{\Delta p_1, \Delta p_2, \ldots \Delta p_k, \ldots\}$$

is a pitch interval set in a pitch system ψ*, then the following function returns the morphetic pitch interval set of* $\underline{\Delta p}$*:*

$$\underline{\Delta\,\mathrm{p}}_{\mathrm{m}}\left(\underline{\Delta p}\right) = \bigcup_{k=1}^{|\underline{\Delta p}|} \{\Delta\,\mathrm{p_m}\left(\Delta p_k\right)\}$$

Definition 751 *If*

$$\underline{\Delta p} = \{\Delta p_1, \Delta p_2, \ldots \Delta p_k, \ldots\}$$

is a pitch interval set in a pitch system ψ*, then the following function returns the frequency interval set of* $\underline{\Delta p}$*:*

$$\underline{\Delta\,\mathrm{f}}\left(\underline{\Delta p}\right) = \bigcup_{k=1}^{|\underline{\Delta p}|} \{\Delta\,\mathrm{f}\left(\Delta p_k\right)\}$$

Definition 752 *If*

$$\underline{\Delta p} = \{\Delta p_1, \Delta p_2, \ldots \Delta p_k, \ldots\}$$

is a pitch interval set in a pitch system ψ*, then the following function returns the chroma interval set of* $\underline{\Delta p}$*:*

$$\underline{\Delta\,\mathrm{c}}\left(\underline{\Delta p}\right) = \bigcup_{k=1}^{|\underline{\Delta p}|} \{\Delta\,\mathrm{c}\left(\Delta p_k\right)\}$$

Definition 753 *If*

$$\underline{\Delta p} = \{\Delta p_1, \Delta p_2, \ldots \Delta p_k, \ldots\}$$

is a pitch interval set in a pitch system ψ*, then the following function returns the morph interval set of* $\underline{\Delta p}$*:*

$$\underline{\Delta\,\mathrm{m}}\left(\underline{\Delta p}\right) = \bigcup_{k=1}^{|\underline{\Delta p}|} \{\Delta\,\mathrm{m}\left(\Delta p_k\right)\}$$

Definition 754 *If*

$$\underline{\Delta p} = \{\Delta p_1, \Delta p_2, \ldots \Delta p_k, \ldots\}$$

is a pitch interval set in a pitch system ψ, then the following function returns the chromamorph interval set of $\underline{\Delta p}$:

$$\underline{\Delta\,\mathrm{q}}\left(\underline{\Delta p}\right) = \bigcup_{k=1}^{|\underline{\Delta p}|} \{\Delta\,\mathrm{q}\,(\Delta p_k)\}$$

Definition 755 *If*

$$\underline{\Delta p} = \{\Delta p_1, \Delta p_2, \ldots \Delta p_k, \ldots\}$$

is a pitch interval set in a pitch system ψ, then the following function returns the chromatic genus interval set of $\underline{\Delta p}$:

$$\underline{\Delta\,\mathrm{g}_\mathrm{c}}\left(\underline{\Delta p}\right) = \bigcup_{k=1}^{|\underline{\Delta p}|} \{\Delta\,\mathrm{g}_\mathrm{c}\,(\Delta p_k)\}$$

Definition 756 *If*

$$\underline{\Delta p} = \{\Delta p_1, \Delta p_2, \ldots \Delta p_k, \ldots\}$$

is a pitch interval set in a pitch system ψ, then the following function returns the genus interval set of $\underline{\Delta p}$:

$$\underline{\Delta\,\mathrm{g}}\left(\underline{\Delta p}\right) = \bigcup_{k=1}^{|\underline{\Delta p}|} \{\Delta\,\mathrm{g}\,(\Delta p_k)\}$$

Deriving *MIPS* interval sets from a chromatic pitch interval set

Definition 757 *If*

$$\underline{\Delta p}_\mathrm{c} = \{\Delta p_{\mathrm{c},1}, \Delta p_{\mathrm{c},2}, \ldots \Delta p_{\mathrm{c},k}, \ldots\}$$

is a chromatic pitch interval set in a pitch system ψ, then the following function returns the chroma interval set of $\underline{\Delta p}_\mathrm{c}$:

$$\underline{\Delta\,\mathrm{c}}\left(\underline{\Delta p}_\mathrm{c}\right) = \bigcup_{k=1}^{|\underline{\Delta p}_\mathrm{c}|} \{\Delta\,\mathrm{c}\,(\Delta p_{\mathrm{c},k})\}$$

Definition 758 *If*

$$\underline{\Delta p}_\mathrm{c} = \{\Delta p_{\mathrm{c},1}, \Delta p_{\mathrm{c},2}, \ldots \Delta p_{\mathrm{c},k}, \ldots\}$$

is a chromatic pitch interval set in a pitch system ψ, then the following function returns the frequency interval set of $\underline{\Delta p}_\mathrm{c}$:

$$\underline{\Delta\,\mathrm{f}}\left(\underline{\Delta p}_\mathrm{c}\right) = \bigcup_{k=1}^{|\underline{\Delta p}_\mathrm{c}|} \{\Delta\,\mathrm{f}\,(\Delta p_{\mathrm{c},k})\}$$

Deriving *MIPS* interval sets from a morphetic pitch interval set

Definition 759 *If*

$$\underline{\Delta p}_\mathrm{m} = \{\Delta p_{\mathrm{m},1}, \Delta p_{\mathrm{m},2}, \ldots \Delta p_{\mathrm{m},k}, \ldots\}$$

is a morphetic pitch interval set in a pitch system ψ, then the following function returns the morph interval set of $\underline{\Delta p}_\mathrm{m}$:

$$\underline{\Delta\,\mathrm{m}}\left(\underline{\Delta p}_\mathrm{m}\right) = \bigcup_{k=1}^{|\underline{\Delta p}_\mathrm{m}|} \{\Delta\,\mathrm{m}\,(\Delta p_{\mathrm{m},k})\}$$

Deriving *MIPS* interval sets from a frequency interval set

Definition 760 *If*

$$\underline{\Delta f} = \{\Delta f_1, \Delta f_2, \ldots \Delta f_k, \ldots\}$$

is a frequency interval set in a pitch system ψ, then the following function returns the chromatic pitch interval set of $\underline{\Delta f}$:

$$\underline{\Delta\,\mathrm{p}_{\mathrm{c}}}\left(\underline{\Delta f}\right) = \bigcup_{k=1}^{|\underline{\Delta f}|} \{\Delta\,\mathrm{p}_{\mathrm{c}}\left(\Delta f_k\right)\}$$

Definition 761 *If*

$$\underline{\Delta f} = \{\Delta f_1, \Delta f_2, \ldots \Delta f_k, \ldots\}$$

is a frequency interval set in a pitch system ψ, then the following function returns the chroma interval set of $\underline{\Delta f}$:

$$\underline{\Delta\,\mathrm{c}}\left(\underline{\Delta f}\right) = \bigcup_{k=1}^{|\underline{\Delta f}|} \{\Delta\,\mathrm{c}\left(\Delta f_k\right)\}$$

Deriving *MIPS* interval sets from a chromamorph interval set

Definition 762 *If*

$$\underline{\Delta q} = \{\Delta q_1, \Delta q_2, \ldots \Delta q_k, \ldots \Delta q_n\}$$

is a chromamorph interval set in a pitch system ψ, then the following function returns the chroma interval set of $\underline{\Delta q}$:

$$\underline{\Delta\,\mathrm{c}}\left(\underline{\Delta q}\right) = \bigcup_{k=1}^{|\underline{\Delta q}|} \{\Delta\,\mathrm{c}\left(\Delta q_k\right)\}$$

Definition 763 *If*

$$\underline{\Delta q} = \{\Delta q_1, \Delta q_2, \ldots \Delta q_k, \ldots \Delta q_n\}$$

is a chromamorph interval set in a pitch system ψ, then the following function returns the morph interval set of $\underline{\Delta q}$:

$$\underline{\Delta\,\mathrm{m}}\left(\underline{\Delta q}\right) = \bigcup_{k=1}^{|\underline{\Delta q}|} \{\Delta\,\mathrm{m}\left(\Delta q_k\right)\}$$

Deriving *MIPS* interval sets from a chromatic genus interval set

Definition 764 *If*

$$\underline{\Delta g_{\mathrm{c}}} = \{\Delta g_{\mathrm{c},1}, \Delta g_{\mathrm{c},2}, \ldots \Delta g_{\mathrm{c},k}, \ldots\}$$

is a chromatic genus interval set in a pitch system ψ, then the following function returns the chroma interval set of $\underline{\Delta g_{\mathrm{c}}}$:

$$\underline{\Delta\,\mathrm{c}}\left(\underline{\Delta g_{\mathrm{c}}}\right) = \bigcup_{k=1}^{|\underline{\Delta g_{\mathrm{c}}}|} \{\Delta\,\mathrm{c}\left(\Delta g_{\mathrm{c},k}\right)\}$$

Deriving *MIPS* interval sets from a genus interval set

Definition 765 *If*

$$\underline{\Delta g} = \{\Delta g_1, \Delta g_2, \ldots \Delta g_k, \ldots\}$$

is a genus interval set in a pitch system ψ, then the following function returns the chromatic genus interval set of $\underline{\Delta g}$:

$$\underline{\Delta\,\mathrm{g}_{\mathrm{c}}}\left(\underline{\Delta g}\right) = \bigcup_{k=1}^{|\underline{\Delta g}|} \{\Delta\,\mathrm{g}_{\mathrm{c}}\left(\Delta g_k\right)\}$$

Definition 766 *If*

$$\underline{\Delta g} = \{\Delta g_1, \Delta g_2, \ldots \Delta g_k, \ldots\}$$

is a genus interval set in a pitch system ψ, then the following function returns the morph interval set of $\underline{\Delta g}$:

$$\underline{\Delta\,\mathrm{m}}\left(\underline{\Delta g}\right) = \bigcup_{k=1}^{|\underline{\Delta g}|} \{\Delta\,\mathrm{m}\left(\Delta g_k\right)\}$$

Definition 767 *If*

$$\underline{\Delta g} = \{\Delta g_1, \Delta g_2, \ldots \Delta g_k, \ldots\}$$

is a genus interval set in a pitch system ψ, then the following function returns the chroma interval set of $\underline{\Delta g}$:

$$\underline{\Delta\,\mathrm{c}}\left(\underline{\Delta g}\right) = \bigcup_{k=1}^{|\underline{\Delta g}|} \{\Delta\,\mathrm{c}\left(\Delta g_k\right)\}$$

Definition 768 *If*

$$\underline{\Delta g} = \{\Delta g_1, \Delta g_2, \ldots \Delta g_k, \ldots\}$$

is a genus interval set in a pitch system ψ, then the following function returns the chromamorph interval set of $\underline{\Delta g}$:

$$\underline{\Delta\,\mathrm{q}}\left(\underline{\Delta g}\right) = \bigcup_{k=1}^{|\underline{\Delta g}|} \{\Delta\,\mathrm{q}\left(\Delta g_k\right)\}$$

4.8.4 Equivalence relations between *MIPS* interval sets

Equivalence relations between pitch interval sets

Equivalence relations between chromatic pitch interval sets

Equivalence relations between morphetic pitch interval sets

Equivalence relations between frequency interval sets

Equivalence relations between chromamorph interval sets

Equivalence relations between chromatic genus interval sets

Equivalence relations between genus interval sets

4.8.5 Inequalities between *MIPS* interval sets

Inequalities between pitch interval sets

Inequalities between chromatic pitch interval sets

Inequalities between morphetic pitch interval sets

Inequalities between frequency interval sets

Inequalities between chroma interval sets

Inequalities between morph interval sets

Inequalities between chromamorph interval sets

Inequalities between chromatic genus interval sets

Inequalities between genus interval sets

4.8.6 Equivalence partitions on *MIPS* interval sets

Equivalence partitions on pitch interval sets

Equivalence partitions on chromatic pitch interval sets

Equivalence partitions on morphetic pitch interval sets

Equivalence partitions on frequency interval sets

Equivalence partitions on chroma interval sets

Equivalence partitions on morph interval sets

Equivalence partitions on chromamorph interval sets

Equivalence partitions on genus interval sets

Theorem 769 *If $\underline{\Delta g}$ is a genus interval set in a pitch system ψ then there exists a unique partition on $\underline{\Delta g}$, called the* morph interval equivalence partition *of $\underline{\Delta g}$ and denoted $\mathrm{P}_{\Delta\,\mathrm{m}}\left(\underline{\Delta g}\right)$, such that*

$$\left(\underline{\Delta g}_1 \in \mathrm{P}_{\Delta\,\mathrm{m}}\left(\underline{\Delta g}\right)\right) \wedge \left(\Delta g_1, \Delta g_2 \in \underline{\Delta g}_1\right) \iff \left(\Delta g_1 \equiv_{\Delta\,\mathrm{m}} \Delta g_2\right)$$

Each element of $\mathrm{P}_{\Delta\,\mathrm{m}}\left(\underline{\Delta g}\right)$ is called a morph interval equivalence class of genus intervals on $\underline{\Delta g}$.

Proof

R1	343	$\Rightarrow$	Morph interval equivalence of genus intervals is an equivalence relation.
R2	R1	$\Rightarrow$	Theorem is proved.

4.8.7 Deriving sets of *MIPS* intervals from sets of *MIPS* objects

Deriving sets of *MIPS* intervals from pitch sets

Deriving sets of *MIPS* intervals from chromatic pitch sets

Deriving sets of *MIPS* intervals from morphetic pitch sets

Deriving sets of *MIPS* intervals from frequency sets

Deriving sets of *MIPS* intervals from chroma sets

Deriving sets of *MIPS* intervals from morph sets

Deriving sets of *MIPS* intervals from chromamorph sets

Deriving sets of *MIPS* intervals from genus sets

Definition 770 *If* $\underline{g}$ *is a genus set in a specified pitch system* ψ *then the* set of genus intervals in $\underline{g}$, *denoted* $\underline{\Delta\,\mathrm{g}}\left(\underline{g}\right)$ *is given by the following formula:*

$$\underline{\Delta\,\mathrm{g}}\left(\underline{g}\right) = \bigcup_{\left(g_1 \in \underline{g}\right)} \bigcup_{\left(g_2 \in \underline{g}\right)} \left\{\Delta\,\mathrm{g}\left(g_1, g_2\right)\right\}$$

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